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Math410: Advanced Calculus I

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Intro & Preliminaries

The Field Axioms

Commutativity of Addition: For all real numbers a and b,

$$a + b = b + a.$$

Associativity of Addition: For all real numbers a, b, and c,

$$(a + b) + c = a + (b + c).$$

The Additive Identity: There is a real number, denoted by 0, such that

$$0 + a = a + 0 = a \qquad \text{for all real numbers } a.$$

The Additive Inverse: For each real number a, there is a real number b such that

$$a + b = 0.$$

Commutativity of Multiplication: For all real numbers a and b,

$$ab = ba.$$

Associativity of Multiplication: For all real numbers a, b, and c,

$$(ab)c = a(bc).$$

The Multiplicative Identity: There is a real number, denoted by 1, such that

$$1a = a1 = a \qquad \text{for all real numbers } a.$$

The Multiplicative Inverse: For each real number $a \neq 0$, there is a real number b such that

$$ab = 1.$$

The Distributive Property: For all real numbers a, b, and c,

$$a(b + c) = ab + ac.$$

The Nontriviality Assumption:

$$1 \neq 0.$$

* See other results from these axioms on pp. 3-4

The Positivity Axioms

There is a set of real numbers, denoted by $\mathcal{P}$, called the set of *positive numbers*. It has the following two properties:

P1 If a and b are positive, then ab and $a + b$ are also positive.

P2 For a real number a, exactly one of the following three alternatives is true:

$$a \text{ is positive}, \qquad -a \text{ is positive}, \qquad a = 0.$$

Interval Notation

For a pair of real numbers a and b such that $a < b$, we define

$$(a, b) \equiv \{x \text{ in } \mathbb{R} \mid a < x < b\},$$
$$[a, b] \equiv \{x \text{ in } \mathbb{R} \mid a \leq x \leq b\},$$
$$(a, b] \equiv \{x \text{ in } \mathbb{R} \mid a < x \leq b\},$$

and

$$[a, b) \equiv \{x \text{ in } \mathbb{R} \mid a \leq x < b\}.$$

Moreover, it is convenient to use the symbols ∞ and $-\infty$ in the following manner. We define

$$[a, \infty) \equiv \{x \text{ in } \mathbb{R} \mid a \leq x\},$$
$$(-\infty, b] \equiv \{x \text{ in } \mathbb{R} \mid x \leq b\},$$
$$(a, \infty) \equiv \{x \text{ in } \mathbb{R} \mid a < x\},$$
$$(-\infty, b) \equiv \{x \text{ in } \mathbb{R} \mid x < b\},$$

and

$$(-\infty, \infty) \equiv \mathbb{R}.$$

The reader should be very careful to observe that although we have defined, say, $[a, \infty)$, we have *not* defined the symbols ∞ and $-\infty$. In particular, we have *not* adjoined additional numbers to $\mathbb{R}$.

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Summary before Final Exam

This course is entirely focused on definitions and proofs. The following is a ***summary*** of key definitions and theorems. It is strongly recommended that you review the precise statements and their corresponding proofs. Be sure to review the first few pages of the class notes to remind yourself of the expectations for proofs, and to familiarize yourself with common errors.

Preliminaries

- Basics of set theory and inequalities
- The **Triangle Inequality** states that $|a+b| \leq |a| + |b|$, for every two real numbers a, b.
- The **Finite Geometric Sum** formula is for $n \in \mathbb{N}$ and any $r \neq 1$, $1 + r + r^2 + \cdots + r^n = \dfrac{1 - r^{n+1}}{1 - r}$.

Chapter 1

- The smallest upper bound of a set S is called its **supremum** and is denoted by $\sup S$. The greatest lower bound of a set S is called its **infimum** and is denoted by $\inf S$.
- The **Completeness Axiom**: Every nonempty set of real numbers S that is bounded above has a supremum.
- The **Archimedean Property** (AP): For every $c \in \mathbb{R}$, and every $\epsilon > 0$ there are $m, n \in \mathbb{N}$ for which $c < m$ and $\dfrac{1}{n} < \epsilon$.
- Let $S \subseteq T \subseteq \mathbb{R}$ be nonempty sets. We say S is **dense** in T if for every $a, b \in T$ with $a < b$, there is an element $s \in S$ for which $a < s < b$.
- $\mathbb{Q}$, and $\mathbb{Q}^c$ are dense in $\mathbb{R}$.

Chapter 2

- To prove $\{a_n\}$ **converges** to a, we need $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ for which $|a_n - a| < \epsilon$ for all $n \geq N$.
- To prove $\{a_n\}$ **diverges to** ∞ if for all $M > 0$, there is $N \in \mathbb{N}$ for which $a_n > M$ for all $n \geq N$.
- The **sum, product, and quotient properties** for convergent sequences are shown.
- The **Comparison Lemma**: Let $\{a_n\}$ converge a. Then $\{b_n\}$ converges to b if there is a nonnegative number C and an index N such that $|b_n - b| \leq C\,|a_n - a| \quad$ for all indices $n \geq N$.
- A sequence $\{a_n\}$ is **bounded** if there is a number M such that $|a_n| \leq M \quad$ for every index n.
- Every convergent sequence is bounded.
- **Sequential Density**: A set S is dense in $\mathbb{R}$ iff every $x \in \mathbb{R}$ is the limit of a sequence in S.
- $S \subset \mathbb{R}$ is **closed** if $\{a_n\}$ is a sequence in S that converges to a number a, then the limit $a \in S$ also.
- $\{a_n\}$ is **monotonically increasing (decreasing)** if $a_{n+1} \geq a_n \quad (a_{n+1} \leq a_n)$ for all n.

- The **Monotone Convergence Theorem** (MCT): A monotone sequence converges iff it is bounded. Moreover, the bounded monotone sequence $\{a_n\}$ converges to $\sup\{a_n \mid n \text{ in } N\}$ ($\inf\{a_n \mid n \text{ in } N\}$) if it is monotonically increasing (decreasing).

- ~~The **Nested Interval Theorem** can be used to show a decreasing sequence of closed intervals share a point.~~

- A set S is called **sequentially compact** if every sequence a_n in S has a subsequence that converges to an element of S.

- The **Sequential Compactness Theorem** (SCT): Every closed and bounded interval $[a, b]$ is sequentially compact (every sequence a_n in $[a, b]$ has a subsequence that converges to an element of $[a, b]$)

Chapter 3

- To prove a function f is **continuous** at x_0: Assume $\{x_n\}$ is a sequence in the domain that approaches x_0. Prove that $\{f(x_n)\} \to f(x_0)$.

- To prove f is not continuous at x_0 we often assume it is and find multiple sequences that approach x_0, use the definition and get a contradiction.

- The **Extreme Value Theorem** (EVT): Any continuous $f : [a, b] \to \mathbb{R}$ attains its maximum/minimum.

- The **Intermediate Value Theorem** (IVT): Suppose $f : [a, b] \to \mathbb{R}$ is continuous. Let c be a number strictly between $f(a)$ and $f(b)$. Then there is a point $x_0 \in (a, b)$ at which $f(x_0) = c$.

- $f : D \to \mathbb{R}$ is **uniformly continuous** if whenever $\{u_n\}$ and $\{v_n\}$ are sequences in D such that $\lim_{n\to\infty} [u_n - v_n] = 0$ then $\lim_{n\to\infty} [f(u_n) - f(v_n)] = 0$

- To prove f is not uniformly continuous, we need to find sequences u_n, v_n for which $(u_n - v_n) \to 0$ but $f(u_n) - f(v_n)$ does not approach zero. This can often be achieved by looking for u_n, v_n where the slope of the graph of f is unbounded.

- Every uniformly continuous function is continuous.

- Every continuous function $f : [a, b] \to \mathbb{R}$ is uniformly continuous.

- $f : D \to \mathbb{R}$ satisfies **the $\epsilon - \delta$ criterion at a point** x_0 in the domain D if for all $\epsilon > 0$ there exists $\delta > 0$ such that for x in D, if $|x - x_0| < \delta$ then $|f(x) - f(x_0)| < \epsilon$. This is equivalent to f being continuous at x_0

- $f : D \to \mathbb{R}$ satisfies **the $\epsilon - \delta$ criterion on the domain** D if for all $\epsilon > 0$ there exists $\delta > 0$ such that for all u, v in D, if $|u - v| < \delta$ then $|f(u) - f(v)| < \epsilon$. This is equivalent to f being uniformly continuous on D.

- If the domain of a strictly monotone function is an interval, then its inverse will be continuous.

- To show $\lim_{x\to x_0} f(x) = \ell$, start with a sequence x_n that converges to x_0, and that $x_n \neq x_0$ for all n, and then prove $\lim_{n\to\infty} f(x_n) = \ell$.

- Typical properties of limits hold: Limit of sum is sum of limits; limit of product is product of limits; and limit of ratio of two functions in ratio of their limits.

Chapter 4: Differentiation

For the entire section, let I be a **neighborhood** of x_0, an open interval $I=(a,b)$ that contains x_0.

- $f: I \to \mathbb{R}$ is **differentiable** at x_0 if $\lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0} \equiv f'(x_0)$ exists.
- Differentiability implies continuity but not the other way around.
- Sums, products, and quotient rules for derivatives are proved.
- **The Chain Rule** states that $(f \circ g)'(x) = f'(g(x))g'(x)$.
- If $y_0 = f(x_0)$ and $f'(x_0) \neq 0$, then $\left(f^{-1}\right)'(y_0) = \frac{1}{f'(x_0)}$.
- **Rolle's Theorem:** Let $f:[a,b] \to \mathbb{R}$ be continuous and $f:(a,b) \to \mathbb{R}$ be differentiable and $f(a)=f(b)$. Then there is a point $c \in (a,b)$ at which $f'(c)=0$.
- **Mean Value Theorem:** Let $f:[a,b] \to \mathbb{R}$ be continuous and $f:(a,b) \to \mathbb{R}$ be differentiable. Then there is a point $c \in (a,b)$ at which $f'(c) = \frac{f(b)-f(a)}{b-a}$
- Let $f: I \to \mathbb{R}$ be differentiable. Then $f: I \to \mathbb{R}$ is constant iff $f'(x)=0$ for all x in I.
- **The Identity Criterion:** Let $g: I \to \mathbb{R}$ and $h: I \to \mathbb{R}$ be differentiable. Then these functions differ by a constant iff $g'(x) = h'(x)$ for all x in I.
- If $f'(x) > 0$, then f is strictly increasing. If $f'(x) < 0$, then f is strictly decreasing.
- **Cauchy Mean Value Theorem:** Given two continuous functions $f, g : [a,b] \to \mathbb{R}$ that are differentiable over (a,b) for which $g'(x) \neq 0$, there is $c \in (a,b)$ for which

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}.$$

- **Corollary to Cauchy Mean Value Theorem:** Assume f is defined over a neighborhood I of x_0 and that $f(x_0) = f'(x_0) = \cdots = f^{(n-1)}(x_0) = 0$. Then, for every point x with $x \neq x_0$, there is a point z that lies strictly between x and x_0 for which

$$f(x) = \frac{f^{(n)}(z)}{n!}\left(x-x_0\right)^n.$$

Chapter 6: Integration

For the entire section, assume $f:[a,b] \to \mathbb{R}$ is bounded and P is a partition of $[a,b]$.

- The **Darboux upper and lower sums** are:

$$L(f,P) \equiv \sum_{i=1}^{n} m_i\left(x_i - x_{i-1}\right) \qquad \text{for } m_i \equiv \inf\left\{f(x) \mid x \in [x_{i-1}, x_i]\right\}$$

$$U(f,P) \equiv \sum_{i=1}^{n} M_i\left(x_i - x_{i-1}\right) \qquad \text{for } M_i \equiv \sup\left\{f(x) \mid x \in [x_{i-1}, x_i]\right\}$$

- For any partition P, $\qquad L(f, P) \leq U(f, P)$.
- Another partition P^* of $[a, b]$ is called a **refinement** of P if each partition point of P is also a partition point of P^*.
- **Refinement Lemma:** For P^*, refinement of P, $L(f, P) \leq L(f, P^*)$ and $U(f, P^*) \leq U(f, P)$.
- **Lower and Upper Integrals:**

$$\underline{\int_a^b} f \equiv \sup\{L(f, P) \mid P \text{ a partition of the interval } [a, b]\}$$

  and

$$\overline{\int_a^b} g \equiv \inf\{U(f, P) \mid P \text{ a partition of the interval } [a, b]\}$$

- $f : [a, b] \to \mathbb{R}$ is **integrable** if $\underline{\int_a^b} f = \overline{\int_a^b} f$.
- **Archimedes-Riemann Theorem:** f is integrable on $[a, b]$ iff there is a sequence $\{P_n\}$ of partitions of the interval $[a, b]$ such that $\lim_{n\to\infty} [U(f, P_n) - L(f, P_n)] = 0$. $\{P_n\}$ is said to be an **Archimedean sequence of partitions**. Moreover,

$$\lim_{n\to\infty} L(f, P_n) = \int_a^b f \qquad \text{and} \qquad \lim_{n\to\infty} U(f, P_n) = \int_a^b f.$$

- Monotone, step functions, and continuous functions are integrable.
- **Additivity of integrals:** If f is integrable, then $\int_a^b f = \int_a^c f + \int_c^b f \quad$ for $c \in (a, b)$.
- **Monotonicity of integrals:** If f and g are integrable with $f \leq g$ then $\int_a^b f \leq \int_a^b g$.
- **Linearity of integrals:** If f, g are integrable on $[a, b]$, then $\int_a^b (\alpha f + \beta g) = \alpha \int_a^b f + \beta \int_a^b g$, for $\alpha, \beta \in \mathbb{R}$.
- If a bounded function f is integrable over every subinterval $[c, d]$ of (a, b) then it is integrable over $[a, b]$ and its integral does not depend on $f(a)$ and $f(b)$.
- **First Fundamental Theorem:** Let $F : [a, b] \to \mathbb{R}$ be continuous and be differentiable on (a, b). Moreover, $F' : (a, b) \to \mathbb{R}$ is continuous and bounded. Then $\int_a^b F'(x)dx = F(b) - F(a)$.
- **Second Fundamental Theorem:** If $f : [a, b] \to \mathbb{R}$ is continuous then $\frac{d}{dx}\left[\int_a^x f\right] = f(x)$ for all $x \in (a, b)$.
- We define $\int_b^a f = -\int_a^b f$ and $\int_c^c f = 0$, where $a < b$.

Chapter 8: Approximation by Taylor Polynomials

For entire section, assume I is a neighborhood of the point x_0 and let n be a nonnegative integer. $f : I \to \mathbb{R}$ and $g : I \to \mathbb{R}$.

- f and g have **contact of order n at x_0** if f and g have n derivatives and $f^{(k)}(x_0) = g^{(k)}(x_0)$ for $0 \le k \le n$.
- The **n-th Taylor polynomial** of $f(x)$ at x_0 is $p_n(x) = \displaystyle\sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k$.
- $f(x) = \displaystyle\sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k$ is called the **Taylor series expansion** for f about the point x_0.
- The **Lagrange Remainder Theorem:** Let f have $n+1$ derivatives. Then for each $x \neq x_0$ in I, $\exists c$ strictly between x and x_0 such that $f(x) = p_n(x) + \dfrac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}$
- An **infinite sum** $\displaystyle\sum_{k=0}^{\infty} a_k$ is defined as $\displaystyle\lim_{n\to\infty} s_n$ for partial sums $s_n = a_1 + \cdots + a_n$.
- **Cauchy Integral Remainder Theorem**: $f(x) = p_n(x) + \dfrac{1}{n!}\displaystyle\int_{x_0}^{x} f^{(n+1)}(t)(x-t)^n \, \mathrm{d}t$, if $f^{(n+1)}$ is continuous on a neighborhood of x_0.

Chapter 9: Sequences and Series of Functions

For notational purposes $\sum a_n = \displaystyle\sum_{n=1}^{\infty} a_n$ is assumed to be a sum of infinite terms.

- $\{a_n\}$ is a **Cauchy sequence** if $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ such that if $n > m \ge N$ then $|a_n - a_m| < \epsilon$.
- Every Cauchy sequence is bounded.
- **Cauchy Convergence Criterion (Sequences)**: A sequence converges iff it is Cauchy.
- Let $\sum a_n$ converge, then $\displaystyle\lim_{n\to\infty} a_n = 0$. The converse is not true but the contrapositive is useful.
- For a number r such that $|r| < 1$, $\displaystyle\sum_{k=0}^{\infty} r^k = \frac{1}{1-r}$.
- Let $\{a_k\}$ be nonnegative. Then the series $\sum a_k$ converges if and only if $\{s_n\}$ is bounded.
- The **Comparison Test**: Suppose that $\{a_k\}$ and $\{b_k\}$ with $0 \le a_k \le b_k$ for all k. Then $\sum a_k$ converges if $\sum b_k$ converges. And $\sum b_k$ diverges if $\sum a_k$ diverges.
- The **Integral Test**: Let $\{a_k\}$ be nonnegative and $f : [1, \infty) \to \mathbb{R}$ is continuous and monotonically decreasing and has the property that $f(k) = a_k$ for every index k. Then the series $\sum a_k$ is convergent iff the sequence of integrals $\left\{\displaystyle\int_1^n f(x)dx\right\}$ is bounded.

- The **Alternating Series Test**: Suppose that $\{a_k\}$ is a monotonically decreasing sequence of non-negative numbers that converges to 0. Then the series $\sum(-1)^{k+1}a_k$ converges.
- **Cauchy Convergence Criterion (Series)** $\sum a_k$ converges iff for all $\epsilon > 0$, $\exists N \in \mathbb{N}$ such that $|a_{n+1} + \cdots + a_{n+k}| < \epsilon$ for all $n \geq N$ and all n$k \in \mathbb{N}$.
- $\sum a_k$ is said to **converge absolutely** if the series $\sum |a_k|$ converges. A series that converges but does not converge absolutely is said to converge conditionally.
- The **Absolute Convergence Test**: $\sum a_k$ converges if $\sum |a_k|$ converges.
- For $\sum a_k$, let $\exists r \in \mathbb{R}$ with $0 \leq r < 1$ and an $N \in \mathbb{N}$ such that $|a_{n+1}| \leq r\,|a_n|$ for all $n \geq N$. Then $\sum a_k$ is absolutely convergent.
- The **Ratio Test for Series**: For the series $\sum a_k$, suppose that $\lim\limits_{n\to\infty} \dfrac{|a_{n+1}|}{|a_n|} = \ell$. If $\ell < 1$, the series $\sum a_k$ converges absolutely. If $\ell > 1$, the series $\sum a_k$ diverges.
- $\{f_n : D \to \mathbb{R}\}$ and $f : D \to \mathbb{R}$
  - $\{f_n\}$ **converges pointwise** to f (write: $f_n \xrightarrow{\text{p.w.}} f$) if for each point x in D, $\forall\epsilon > 0$, $\exists N \in \mathbb{N}$ such that $\forall n \geq N$ then $|f(x) - f_n(x)| < \epsilon$
  - $\{f_n\}$ **converge uniformly** to f (write: $f_n \xrightarrow{u} f$) if $\forall\epsilon > 0$, $\exists N \in \mathbb{N}$ such that $\forall n \geq N$ and $\forall x \in D$ then $|f(x) - f_n(x)| < \epsilon$
  - $\{f_n\}$ is **uniformly Cauchy** if $\forall\epsilon > 0$, $\exists N \in \mathbb{N}$ such that $\forall n, m \geq N$ and $\forall x \in D$ then $|f_n(x) - f_m(x)| < \epsilon$
- The **Weierstrass Uniform Convergence Criterion**: $f_n \xrightarrow{u} f$ iff $\{f_n\}$ is uniformly Cauchy.
- If a sequence of functions is continuous (integrable) its uniform limit is continuous (integrable).
- The uniform limit of a sequence of differentiable functions may not be differentiable.
- $f : I \to \mathbb{R}$ is **continuously differentiable** if it is differentiable and its derivative is continuous.
- Let $\{f_n : I \to \mathbb{R}\}$ be continuously differentiable functions with $f_n \xrightarrow{\text{p.w.}} f$ and$f'_n \xrightarrow{u} g$. Then the function $f : I \to \mathbb{R}$ is continuously differentiable, and $f'(x) = g(x)$ for all x in I.
- Given a sequence $\{c_k\}$, the **domain of convergence** of the series $\sum\limits_{k=0}^{\infty} c_k x^k$ to be the set of all numbers x such that the series $\sum\limits_{k=0}^{\infty} c_k x^k$ converges. We then define a function $f : D \to \mathbb{R}$ by $f(x) = \lim\limits_{n\to\infty}\left[\sum\limits_{k=0}^{n} c_k x^k\right] = \sum\limits_{k=0}^{\infty} c_k x^k$ for all x in D as the **power series expansion**.
- Let $x_0 > 0$ be in the domain of convergence of the power series $\sum\limits_{k=0}^{\infty} c_k x^k$. Let $0 < r < |x_0|$. Then $[-r, r]$ is in the domain of convergence of both $\sum\limits_{k=0}^{\infty} c_k x^k$ and $\sum\limits_{k=1}^{\infty} k c_k x^{k-1}$. Moreover, both $\sum\limits_{k=0}^{\infty} c_k x^k$ and $\sum\limits_{k=1}^{\infty} k c_k x^{k-1}$ converge uniformly on $[-r, r]$.

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Guide to Proof Writing

A proof is an argument designed to convince your audience that a mathematical statement is true. It may consist of calculations, a verbal explanation, or a mix of both. Unlike typical computational math problems, writing a proof demands a higher level of mathematical rigor, clarity, and effective communication. A proof is a sequence of statements, each logically following from the one before, to arrive at the conclusion, relying solely on the hypotheses, definitions, and known true statements.

Example of a Theorem and Proof

Theorem 1 Let a and b be real numbers. Suppose that $0 < a \leq b$. Then $\sqrt{a} \leq \sqrt{b}$. Note that, for a nonnegative real number x, $\sqrt{x}$ denotes the nonnegative square root of x.

Proof. Since $a \leq b$ by assumption, it follows that

$$(b - a) \geq 0.$$

We know that a and b are both positive, so their square roots are defined. Thus, we can use a difference of squares expansion to write:

$$(\sqrt{b} - \sqrt{a})(\sqrt{b} + \sqrt{a}) = (b - a) \geq 0.$$

Since a and b are strictly positive, so are $\sqrt{a}$ and $\sqrt{b}$. Thus $(\sqrt{b} + \sqrt{a}) > 0$, and we can divide through the above expression without changing the direction of the inequality. We get:

$$(\sqrt{b} - \sqrt{a}) = \frac{(b - a)}{(\sqrt{b} + \sqrt{a})} \geq \frac{0}{(\sqrt{b} + \sqrt{a})} = 0$$

and so we conclude that $\sqrt{b} \geq \sqrt{a}$.

$\blacksquare$

There is some discretion involved in writing proofs in deciding which statements are well-known or self evident and which need further justification. For example, in the above proof, we used the fact that positive numbers have positive square roots without justification.

Example of a False Statement and Counterexample

To prove that a statement is true, you must show that the conclusion holds in every situation that satisfies the hypotheses. You can, however, prove that a statement is false by finding a single instance where the hypotheses hold but the conclusion fails. This is called a counterexample to the statement.

False Statement. For all real numbers a and b, $|a + b| = |a| + |b|$.

Counterexample. Consider the example $a = 1, b = -1$. Then

$$|a + b| = |0| = 0 \quad \text{but} \quad |a| + |b| = 1 + 1 = 2.$$

It is therefore not always true that $|a + b| = |a| + |b|$.

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Common Proof Mistakes to Avoid

Common Mistake #1. A written proof starting with the conclusion, and working backwards toward the hypotheses.

This approach can be useful for brainstorming how the proof might work, but it is not suitable for the final presentation of the proof. At best, such reasoning results in poor exposition, and at worst, it can lead to logical errors. Always begin a proof with the hypotheses and known truths, and then work towards the conclusion.

Common Mistake #2. Testing the conclusion with a few specific examples and inferring that it always holds.

A proof must provide a rigorous argument demonstrating that the conclusion is valid in every case where the hypotheses are met.

Common Mistake #3. Assuming the conclusion.

Starting a proof by assuming the statement to be proved as true, rather than deriving it from known facts or hypotheses.

To prove that two expressions A and B are equal, avoid starting by assuming $A = B$ and manipulating both sides. This method is invalid and can lead to false conclusions. Instead, choose one side and work through equalities to reach the other side.

Common Mistake #4. Overlooking special cases.

Failing to consider special or boundary cases that could affect the validity of the proof. For instance, if a step in the proof requires dividing by an unknown real number x, you must separately consider the case where $x = 0$.

Common Mistake #5. Undeclared variables and undefined notation.

A symbol like x has no meaning until it is given a definition. A proof should explicitly declare all the objects involved, such as "let x be a real number" or "let f be a continuous function from $\mathbb{R}$ to $\mathbb{R}$". Likewise, any nonstandard notation or shorthand should be clearly defined.

Common Mistake #6. Taking steps in the proof without sufficient justification.

In this course, justify any non-obvious statements and cite results from the course, verifying their hypotheses. Your proof should be detailed enough for a student in this course to understand. Ensure the proof's structure is clear: explain how statements and computations connect and lead to the conclusion. Use full English sentences to communicate your argument effectively.

Common Mistake #7. Mixing up quantifiers.

Incorrectly applying quantifiers like "for all" ($\forall$) and "there exists" ($\exists$), leads to logical inconsistencies or misinterpretations.

Common Mistake #8. Incorrect use of proof techniques.

Misapplying proof techniques like proof by contradiction, induction, or contraposition, leading to invalid conclusions.

When using a proof by contradiction, verify you identify the correct statement that represents the logical contradiction. The contradiction of A implies B is that A is true and B is false at the same time.

Common Mistake #9. Imprecise statements.

An argument cannot be considered rigorous if it includes ambiguous or poorly defined statements. Be precise, use mathematical terminology accurately, and ensure that it is applied correctly.

The statement "The function has a maximum somewhere" is vague. A clearer statement would be: "The function f has a maximum at $x = c$ if $f(c) \geq f(x)$ for all x in the domain of f." This specifies that $f(c)$ is the maximum value and is greater than or equal to all other values of f within its domain.

Common Mistake #10. General bad communication.

If you were writing an essay, you wouldn't create your own words, symbols, or shorthand, or present fragmented ideas and chaotic thoughts. Apply the same rigor to your proofs. Your goal is to communicate clearly and logically. Ensure your proof is neat, grammatically correct, and follows a logical order, using textbook proofs as models.

Re-read your proofs to check if all terms and symbols are defined, if the order is logical, and if assumptions, statements, and deductions are clear. Ensure each line's purpose is evident and that the proof either progresses from known facts to the conclusion or from one side of an equation through equalities to the other side. Having someone else review your work can also help identify areas needing clarification.

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Common Proof Strategies

* Write out exactly what you want to show using definitions!
- cases of even/odd/neg/pos when working with integers
- add and subtract something to introduce what I need
- when working with continuity, start with a sequence in the domain of f that converges
- For a monotonically increasing sequence, a_1 serves as a lower bound!
- For a subsequence $\{a_{n_k}\}$ of $\{a_n\}$, the following follows from the definition of a subsequence:

  For every k, there exists an n such that $(a_n)_k = a_n$.

  (and $n_k \geq k$)
- Do we prove by using a definition (usually more straightforward), or using another theorem?
- Look at the problem, think, then come back to it!

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- Just as important as writing out what you know, write out what you don't know to see what you have ~~freedom~~ to choose (or make cases out of)!
- Completely stuck or frustrated? Prove if $a<b$, then $a<\frac{a+b}{2}<b$.
- Introduce what I need, then find the justification for the proof (e.g. see proof that set of dyadic numbers $\frac{k}{2^n}$ $\begin{matrix} k \to \in \mathbb{Z} \\ n \to \in \mathbb{N} \end{matrix}$ are dense)
- Write out the definition of what you want to prove, play with examples/numbers, but you can't rely on just one of these approaches!
- Don't know limit but want to show convergence? Think Cauchy!

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·Use formulas that seem relevant, but they may simplify based on givens in the question

(10 pts) Let I be a neighborhood of a real number x_0 and let n be a natural number. Suppose $f : I \to \mathbb{R}$ has $n+1$ derivatives and that $f^{(n+1)} : I \to \mathbb{R}$ is continuous. Assume $f^{(k)}(x_0) = 0$ for all $k, 1 \leq k \leq n$. Suppose $f^{(n+1)}(x_0) < 0$ and n is odd. Prove that x_0 is a local maximizer for f.

The Completeness Axiom (1.4) / Inequalities and Identities (1.3)

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<u>WT:</u>

Restate the following from memory:

(a) Bounded Set (above) - A non-empty set S is bounded if and only if (iff) there exists some $c \in \mathbb{R}$ such that for all $x \in S$, we have that $x \leq c$. upper bound

$(\exists c \in \mathbb{R})(\forall x \in S)$

(b) Completeness Axiom - For a non-empty set S that is bounded above, there is a least upper bound among all the upper bounds of S, which we call the supremum (sup).

*Note: If the $\sup(S) \in S$, then $\sup(S)$ is the "maximum" of S

(c) Triangle Inequality - For real numbers a and b (i.e., $a, b \in \mathbb{R}$), we have that

$$|a+b| \leq |a| + |b|$$

Proof Recall:

Theorem 1.4 *Suppose that S is a nonempty set of real numbers that is bounded below. Then among the set of lower bounds for S there is a largest, or greatest, lower bound.*

↳ infimum, or inf(S)

Proof. We will consider the set obtained by "reflecting" the set S about the number 0; that is, we will consider the set

symbol for "define"

$$T \equiv \{x \in \mathbb{R} \mid \underline{\quad -x \quad} \in S\}.$$

For any number x,

$$b \leq x \quad \text{if and only if} \quad \underline{\quad -x \quad} \leq -b.$$

Thus, a number b is a lower bound for S if and only if the number $\underline{\quad -b \quad}$ is an upper bound for T.
Since the set S has been assumed to be bounded below, it follows that the set T is <u>bounded above</u>.
The Completeness Axiom asserts that there is a <u>least upper bound</u> for T, which we denote by c.
Since lower bounds of S occur as negatives of upper bounds for T, the number $\underline{\quad -c \quad}$ is the greatest lower bound for S. ■

The Distribution of The Integers and Rationals (1.2)

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WT:

(a) Archimedian Property -

(i) For any positive number c, there is some natural number n such that $n > c$.

$\mathbb{N}$ (ii) For any positive number ε, there is some natural number n such that $\frac{1}{n} < \varepsilon$.

(b) Dense — A set S is dense provided that for any interval (a,b), with $a<b$, there is an element $x \in S$ such that $x \in (a,b)$.

* non-example: $\mathbb{Z}$ → "integers"

$(1.1, 1.2)$

* example: $\mathbb{Q}$ → "rationals"

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Proof Recall:

$\frac{a \in \mathbb{Z}}{b \in \mathbb{Z}, b \neq 0}$

SCRAP

$a < \frac{m}{n} < b$

$an < m < bn$

Theorem 1.9 *The set of rational numbers is dense in* $\mathbb{R}$.

$0 < b - a$

Proof. Let a and b be real numbers such that $a < b$. We need to show that the interval (a, b) contains a rational number. By the Archimedean Property, we can choose a natural number n such that

$a < b - \frac{1}{n} < b$

$$\frac{1}{n} < \underline{b - a},$$

For any number c, there is exactly one integer in the interval $[c, c+1)$

so $\frac{1}{n}$ is less than the length of the interval (a, b). By $\boxed{\text{Theorem 1.8}}$ applied to $c \equiv nb - 1$, there is an integer m in the interval $\underline{[nb-1, nb)}$. Thus,

"define"

$$\underline{nb - 1} \leq m < \underline{nb},$$

which, after dividing by n, gives

$$\underline{b - \frac{1}{n}} \leq \frac{m}{n} < \underline{b}. \qquad (1.4)$$

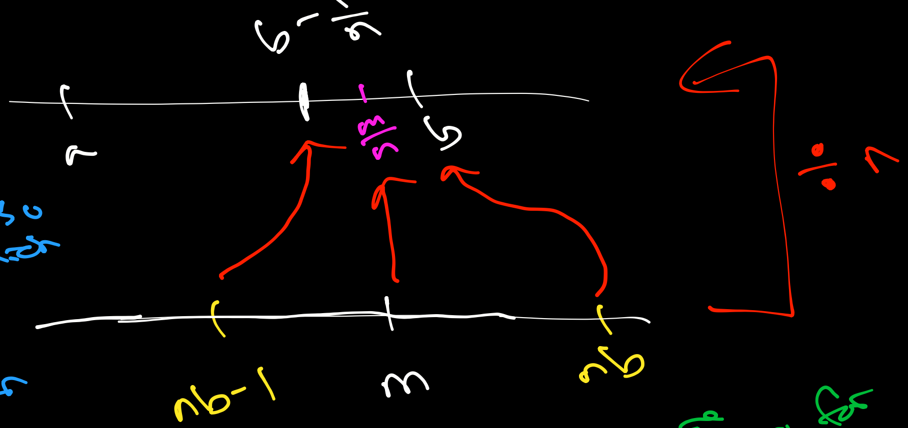
But $\frac{1}{n} < b - a$, so

$$a = b - (b - a) < \underline{b - \frac{1}{n}}. \qquad (1.5)$$

$a < \frac{m}{n} < b$

From the inequalities (1.4) and (1.5) we conclude that the rational number $\frac{m}{n}$ belongs to the interval $\underline{(a, b)}$. ■

$* \ \frac{1}{n} < b - a \Longleftrightarrow b - \frac{1}{n} > a$



*Chose n so $a < b - \frac{1}{n}$, which we can do by the Archimedean Property (with $b - a > 0$)

*note we are assuming $b > 0$ for the sake of the diagram

The Convergence of Sequences (2.1)

WT:

(a) Convergence of a Sequence -

A sequence $\{a_n\}$ is said to converge to a if and only if the following is true:

For all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $|a_n - a| < \varepsilon$ for all indices $n \geq N$.

ex) $\{a_n\} = \frac{2n+1}{3n-2}$. Find $\lim_{n \to \infty} \frac{2n+1 \cdot \frac{1}{n}}{3n-2 \cdot \frac{1}{n}} = \boxed{\frac{2}{3}}$

(b) Comparison Lemma -

Let $\{a_n\}$ converge to a. Then $\{b_n\}$ converges to b if there is a nonnegative number C and an index N_* such that

$$|b_n - b| \leq C|a_n - a|$$

for all $n \geq N_*$

ex) $\frac{1}{n} \to 0$, $\frac{2}{n} \to 0$ $\quad \left|\frac{1}{n} - 0\right| \leq 1\left|\frac{2}{n} - 0\right|$ C

ex) $\left|\frac{2n}{n+1} - 2\right| \leq 2\left|\frac{n}{n+1} - 1\right|$

(c) Polynomial Property of Convergent Sequences -

If $\{a_n\}$ converges to a, then for any polynomial $p: \mathbb{R} \to \mathbb{R}$, $\lim_{n \to \infty} p(a_n) = p(a)$.

(c) Divergence -

For every a, there exists $\varepsilon > 0$ such that for every N, there is some $n \geq N$ with $|a_n - a| \geq \varepsilon$.

negate!

convergence

For every $\varepsilon > 0$, there exists some N such that, for all $n \geq N$, $|a_n - a| < \varepsilon$.

---

ex) $\{n^2\}$

SCRAP

$|n^2 - L| < \varepsilon$

*Note that $\{n^2\} \geq 0$ for all n, so $L \geq 0$

$-\varepsilon < n^2 - L < \varepsilon$

Lemma 2.21 Suppose that the sequence $\{d_n\}$ converges to the number d and that $d_n \geq 0$ for every natural number n. Then $d \geq 0$.

$-\varepsilon + L < n^2 < \varepsilon + L$

$n^2 \geq 0$

$0 \leq n^2 < \varepsilon + L$

$n < \sqrt{\varepsilon + L}$

Proof: Suppose $\{a_n\} = n^2$ converges to L. Then by definition of convergence, for all $\varepsilon > 0$, there exists N such that if $n \geq N$, then $|n^2 - L| < \varepsilon$. By the A.P., we'll choose $N > \sqrt{\varepsilon + L}$, so $n \geq N > \sqrt{\varepsilon + L}$, or

$n > \sqrt{\varepsilon + L}$. Then $|n^2 - L| > \varepsilon$ for all $n \geq N$, a contradiction! ∎

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ex)

Prove $\{n^2+2n-3\}$ diverges to ∞.

Scrap

$(n-1)(n+3)$

Assume $n-1>1$ (AP), or $n>2$

Then $(n-1)(n+3)>(1)(n+3)$

Assume $n+3>M$, or $n>M-3$.

Then, with both assumptions, we have that $(n-1)(n+3)>(1)M$

Now take $N>\max(2, M-3)$

Proof:

It suffices to show that there exists some N such that $a_n>M$ for all $M>0$ and $n\geq N$. By the Archimedean Property, we can select $N>\max(2, M-3)$. Then if $n\geq N$, we have the following:

$$\begin{aligned} n^2+2n-3 &= (n-1)(n+3) \\ &> (1)(M) \\ &= M \end{aligned}$$

Therefore, a_n diverges to ∞ by definition. $\blacksquare$

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ex) Prove that $\left\{ \frac{4n-1}{2n} \right\}$ converges to 2.

SCRAP

$$\left| \frac{4n-1}{2n} - 2 \right| < \varepsilon$$

$$\left| \frac{4n-1-2(2n)}{2n} \right| < \varepsilon$$

$$\left| \frac{-1}{2n} \right| < \varepsilon$$

$\leftarrow |-1|$

$$\frac{1}{2n} < \varepsilon$$

$2n \leftarrow$ already pos!

$$2n > \frac{1}{\varepsilon}$$

$$n > \frac{1}{2\varepsilon}$$

Proof:

Assume $\varepsilon > 0$. Moreover, assume $N > \frac{1}{2\varepsilon}$ (such an N exists by the Archimedean Property), and $n \geq N$. Observe the following:

$$n > \frac{1}{2\varepsilon}$$

$$\frac{1}{2n} < \varepsilon$$

$*|1| = |-1|$

$$\left| \frac{-1}{2n} \right| < \varepsilon$$

$$\left| \frac{4n-1}{2n} - 2 \right| < \varepsilon$$

Therefore, by the def. of convergence $\left\{ \frac{4n-1}{2n} \right\}$ converges to 2.

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Proof Recall:

Theorem 2.10 (The Sum Property) *Suppose that the sequence $\{a_n\}$ converges to the number a and that the sequence $\{b_n\}$ converges to the number b. Then the sequence of sums $\{a_n + b_n\}$ converges to the sum $a + b$; that is,*

$$\lim_{n\to\infty} [a_n + b_n] = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n.$$

Proof. Let $\varepsilon > 0$. We need to find an index N such that

$$|(a_n + b_n) - (a + b)| < \underline{\varepsilon} \quad \text{for all indices } n \geq N. \tag{2.7}$$

In order to do so, we first observe that for every index n,

$$|(a_n + b_n) - (a + b)| = |(a_n - a) + \underline{b_n - b}|,$$

and hence, by the Triangle Inequality,

$$|(a_n + b_n) - (a + b)| \leq |a_n - a| + \underline{|b_n - b|}. \tag{2.8}$$

Since the sequence $\{a_n\}$ converges to a and $\varepsilon/2$ is positive, we can choose an index N_1 such that

-or- $|a_n - a| < \varepsilon_*$ $|a_n - a| < \ell_*$, where $\ell_* = \frac{\varepsilon_*}{2}$

$$|a_n - a| < \underline{\tfrac{\varepsilon}{2}} \quad \text{for all indices } n \geq N_1,$$

and since the sequence $\{b_n\}$ converges to b and $\varepsilon/2$ is positive, we can choose an index N_2 such that

$$|b_n - b| < \underline{\tfrac{\varepsilon}{2}} \quad \text{for all indices } n \geq N_2.$$

Define $N \equiv \max\{N_1, N_2\}$. From inequality (2.8) and the choice of N_1 and N_2, it follows that if $n \geq N$, then

$$|(a_n + b_n) - (a + b)| \leq |a_n - a| + |b_n - b| < \underline{\tfrac{\varepsilon}{2}} + \underline{\tfrac{\varepsilon}{2}} = \varepsilon.$$

Thus, the required inequality (2.7) holds for this choice of N. ■

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Sequences And Sets (2.2)

(a) Bounded Sequence - $\{a_n\}$ is bounded iff there exists M such that...

$$|a_n| \le M \text{ for all } n$$

$$-M \le a_n \le M$$

(b) Every convergent sequence is bounded.

(c) Proposition 2.19:

A set S is dense in $\mathbb{R}$ iff every number $x \in \mathbb{R}$ is the limit of a sequence in S.

(d) Sequential Density of $\mathbb{Q}$ - Every number is the limit of a sequence of rational numbers.

(e) Closed - A subset S of $\mathbb{R}$ is closed provided that if $\{a_n\}$ is a sequence in S that converges to a, then a belongs to S.

The Monotone Convergence Theorem (2.3)

(a) Monotonically increasing:

Definition A sequence $\{a_n\}$ is said to be *monotonically increasing* provided that

$$a_{n+1} \geq a_n \qquad \text{for every index } n.$$

A sequence $\{a_n\}$ is said to be *monotonically decreasing* provided that

$$a_{n+1} \leq a_n \qquad \text{for every index } n.$$

A sequence $\{a_n\}$ is called *monotone* if it is either monotonically increasing or monotonically decreasing.

(b) Monotone Convergence Theorem:
A monotone sequence converges iff it is bounded.

Theorem 2.25 The Monotone Convergence Theorem A monotone sequence converges if and only if it is bounded. Moreover, the bounded monotone sequence $\{a_n\}$ converges to

i. $\sup\{a_n \mid n \text{ in } N\}$ if it is monotonically increasing, and to
ii. $\inf\{a_n \mid n \text{ in } N\}$ if it is monotonically decreasing.

(c) Let c be a number with $|c| < \underline{1}$.

Then $\lim_{n\to\infty} c^n = \underline{0}$.

The Sequential Compactness Theorem (2.4)

Definition Consider a sequence $\{a_n\}$. Let $\{n_k\}$ be a sequence of natural numbers that is strictly increasing; that is,

$\hookrightarrow = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\}$

$$n_1 < n_2 < n_3 < \cdots.$$

indices!

Then the sequence $\{b_k\}$ defined by

$(a_n)_k = \{\frac{1}{3}, \frac{2}{4}, \dots\}$

$$b_k = a_{n_k} \qquad \text{for every index } k$$

is called a *subsequence* of the sequence $\{a_n\}$.

Often a subsequence of $\{a_n\}$ is simply denoted by $\{a_{n_k}\}$, it being implicitly understood that $\{n_k\}$ is a strictly increasing sequence of natural numbers and that the kth term of the sequence $\{a_{n_k}\}$ is a_{n_k}.[4]

WT-

(a) Consider the sequence $\{n^2\}$ $\{a_n\}$.

Is the sequence $\{1, 4, 1, 4, \dots\}$ $(a_n)_k$ a subsequence of $\{n^2\}$? Why or why not?

$a_n: \{1, 4, 9, 16, \dots\}$

a_1, a_2, a_3, a_4

$(a_n)_k = \{1, 4, 1, 4, \dots\}$

$(a_n)_1, (a_n)_2, (a_n)_3$

(b) The text mentions, for a sequence a_n and subsequence a_{n_k} (or $(a_n)_k$) that $n_k \geq k$. What's the intuition here?

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Proposition 2.30 Let the sequence $\{a_n\}$ converge to the limit a. Then every subsequence of $\{a_n\}$ also converges to the same limit a.

Theorem 2.32 Every sequence has a monotone subsequence.

* **Theorem 2.33** Every bounded sequence has a convergent subsequence.

→ Follows from Thm 2.32 and Monotone Convergence Thm.

Definition A set of real numbers S is said to be *sequentially compact* provided that every sequence $\{a_n\}$ in S has a subsequence that converges to a point that belongs to S.

Theorem 2.36 The Sequential Compactness Theorem Let a and b be numbers such that $a < b$. Then the interval $[a, b]$ is sequentially compact; that is, every sequence in $[a, b]$ has a subsequence that converges to a point in $[a, b]$.

→ i.e., any closed interval is sequentially compact!

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A subtle note...

Proposition 2.30 Let the sequence $\{a_n\}$ converge to the limit a. Then every subsequence of $\{a_n\}$ also converges to the same limit a.

For all k, $(a_n)_k = a_n$ for some n.

Proof

Let $\epsilon > 0$. We need to find an index N such that

$$|a_{n_k} - a| < \epsilon \qquad \text{for all indices } k \geq N. \tag{2.22}$$

Since the whole sequence $\{a_n\}$ converges to a, we can choose an index N such that

$$|a_n - a| < \epsilon \qquad \text{for all indices } n \geq N. \tag{2.23}$$

But observe that since $\{n_k\}$ is a strictly increasing sequence of natural numbers,

$$n_k \geq k \qquad \text{for every index } k.$$

Thus, the required inequality (2.22) follows from inequality (2.23). ∎

What is this saying?

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Proof Recall

* **Definition** A set of real numbers S is said to be *sequentially compact* provided that every sequence $\{a_n\}$ in S has a subsequence that converges to a point that belongs to S.

Theorem 2.36 (The Sequential Compactness Theorem): *Let a and b be numbers such that $a < b$. Then the interval $[a, b]$ is sequentially compact; that is, every sequence in $[a, b]$ has a subsequence that converges to a point in $[a, b]$.*

i.e, not all subsequences in $[a,b]$ converge

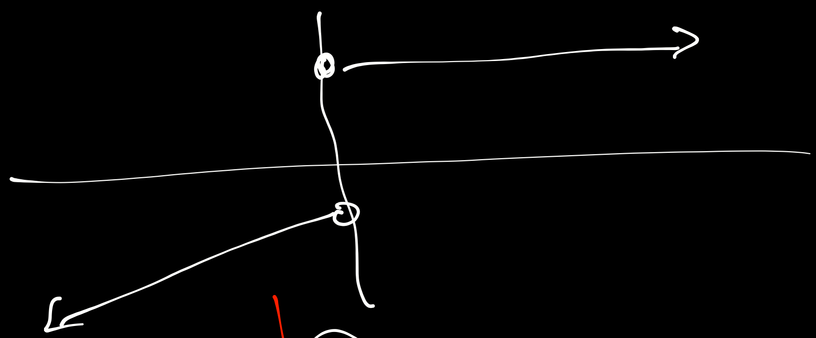
Proof:
There are two distinct parts to the proof. First, it is necessary to show that a sequence in $[a, b]$ has a convergent subsequence. Then it must be shown that the limit of this subsequence belongs to the interval $[a, b]$.
Let x_n be a sequence in $[a, b]$. Then x_n is bounded. Hence, by the preceding theorem, there is a subsequence x_{n_k} that converges.
But the sequence x_{n_k} is a sequence in $[a,b]$, and hence, according to Theorem 2.22, its limit is also in $[a,b]$. ■

* **Theorem 2.22** Let $\{c_n\}$ be a sequence in the interval $[a, b]$. If $\{c_n\}$ converges to the number c, then c also belongs to the interval $[a, b]$.

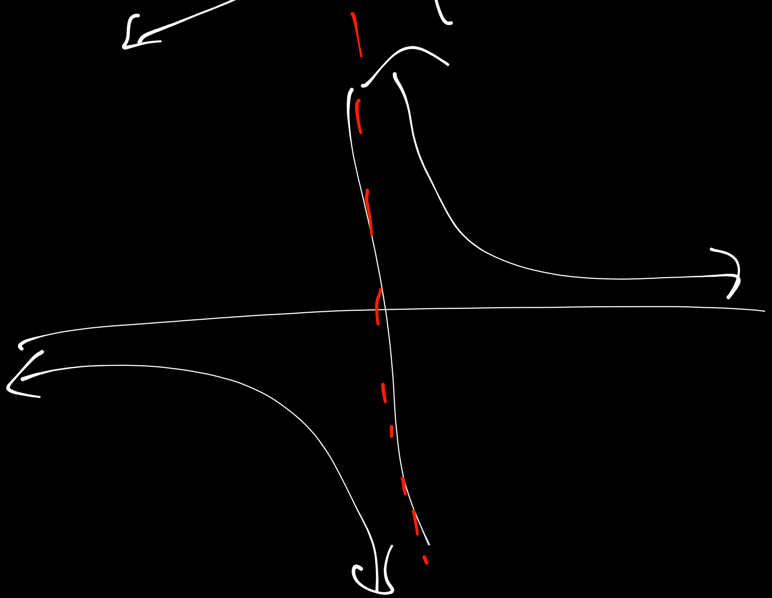
Continuity (3.1)

WU - Brainstorm with a partner different "ways" a function could be discontinuous (maybe using a graph).

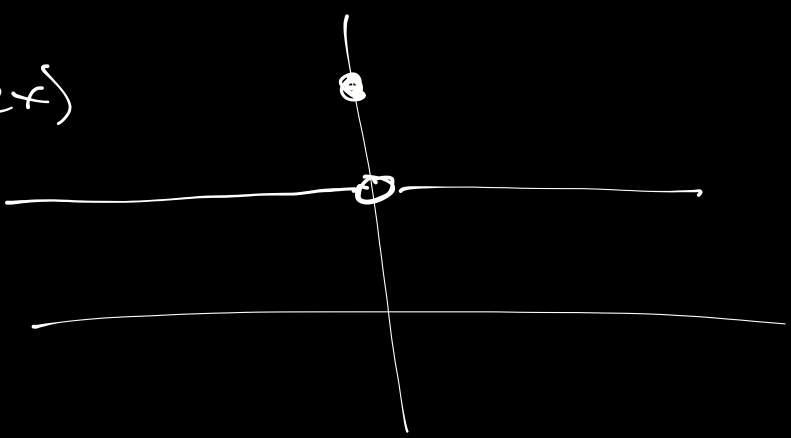
ex)



ex)



ex)



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Continuity (3.1)

(a) Continuous (in terms of sequences) -

function with domain D → mapping g^{π} → codomain

Definition A function $f\colon D \to \mathbb{R}$ is said to be *continuous at the point* x_0 in D provided that whenever $\{x_n\}$ is a sequence in D that converges to x_0, the image sequence $\{f(x_n)\}$ converges to $f(x_0)$. The function $f\colon D \to \mathbb{R}$ is said to be *continuous* provided that it is continuous at every point in D.

i.e., $\lim_{n\to\infty} f(x_n) = f(x_0)$

(b) Proof Recall -

Theorem 3.6: *For functions $f : D \to \mathbb{R}$ and $g : U \to \mathbb{R}$ such that $f(D)$ is contained in U, suppose that $f : D \to \mathbb{R}$ is continuous at the point x_0 in D and $g : U \to \mathbb{R}$ is continuous at the point $f(x_0)$. Then the composition $g \circ f : D \to \mathbb{R}$ is continuous at x_0.*

Proof:

Let x_n be a sequence in D that converges to x_0. By the continuity of the function $f : D \to \mathbb{R}$ at the point x_0, the sequence $f(x_n)$ converges to $f(x_0)$. But then $f(x_n)$ is a sequence in U that converges to $f(x_0)$, so by the continuity of $g : U \to \mathbb{R}$ at the point $f(x_0)$, the sequence $g(f(x_n))$ converges to $g(f(x_0))$; that is, $\lim_{n\to\infty}(g \circ f)(x_n) = (g \circ f)(x_0)$.

Thus, the composition $g \circ f$ is continuous at x_0. ■

Example 3.3 Define the function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

This function is called *Dirichlet's function*. There is no point x_0 in $\mathbb{R}$ at which Dirichlet's function is continuous. Indeed, given a point x_0 in $\mathbb{R}$, by the sequential density of the rationals and irrationals (recall Theorem 2.20), there is a sequence $\{u_n\}$ of rational numbers that converges to x_0 and also a sequence $\{v_n\}$ of irrational numbers that converges to x_0. But $\{f(u_n)\}$ is a constant sequence all of whose terms equal 1 while $\{f(v_n)\}$ is a constant sequence all of whose terms equal 0. Thus,

$$\lim_{n\to\infty} f(u_n) = 1 \neq 0 = \lim_{n\to\infty} f(v_n).$$

Since both of the sequences $\{u_n\}$ and $\{v_n\}$ converge to x_0, it is not possible for f to be continuous at x_0. Observe that one expression of the discontinuous nature of Dirichlet's function is that there is no way to graph it. ■

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Exam #1

Review

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Problems

This course is entirely focused on definitions and proofs. The best practice for the exam is reviewing definitions and theorems done in class, homework problems, exploring additional exercises in the textbook. Be sure to review the first few pages of the class notes to remind yourself of the expectations for proofs, and to familiarize yourself with common errors.

1. Suppose a_n is a bounded sequence, and b_n is a sequence that diverges to ∞. Prove that $a_n + b_n$ diverges to ∞.

2. Using the definition of limit, find the limit of the sequence $a_n = \dfrac{2n+1}{n-1} + \dfrac{1}{n}$.

3. Prove that a monotone sequence is bounded if it has a bounded subsequence.

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Solutions

1. Suppose a_n is a bounded sequence, and b_n is a sequence that diverges to ∞. Prove that $a_n + b_n$ diverges to ∞.

Solution:

$-M \leq a_n \leq M$

Proof: By definition there is a real number M for which $|a_n| \leq M$ for all $n \in \mathbb{N}$. Let A be a positive real number. Since $b_n \to \infty$, there is $N \in \mathbb{N}$ for which $b_n > A + M$ for all $n \geq N$. Therefore, for all $n \geq N$

$$a_n + b_n > a_n + A + M \geq -M + A + M = A \Rightarrow a_n + b_n > A.$$

This means $a_n + b_n \to \infty$, as desired.

■

2. Using the definition of limit, find the limit of the sequence $a_n = \dfrac{2n+1}{n-1} + \dfrac{1}{n}$.

Solution:

Proof: Let $\epsilon > 0$. By the Archimedean Property, there is $N \in \mathbb{N}$ for which $N > 1 + \frac{4}{\epsilon}$. Therefore $N - 1 > \frac{4}{\epsilon}$, which implies $\frac{4}{N-1} < \epsilon$. If $n \geq N$, then

$$\left|\frac{2n+1}{n-1} + \frac{1}{n} - 2\right| = \frac{3}{n-1} + \frac{1}{n} < \frac{4}{n-1} \leq \frac{4}{N-1} < \epsilon$$

Therefore, by the definition of limit, $\lim\limits_{n\to\infty} a_n = 2$.

■

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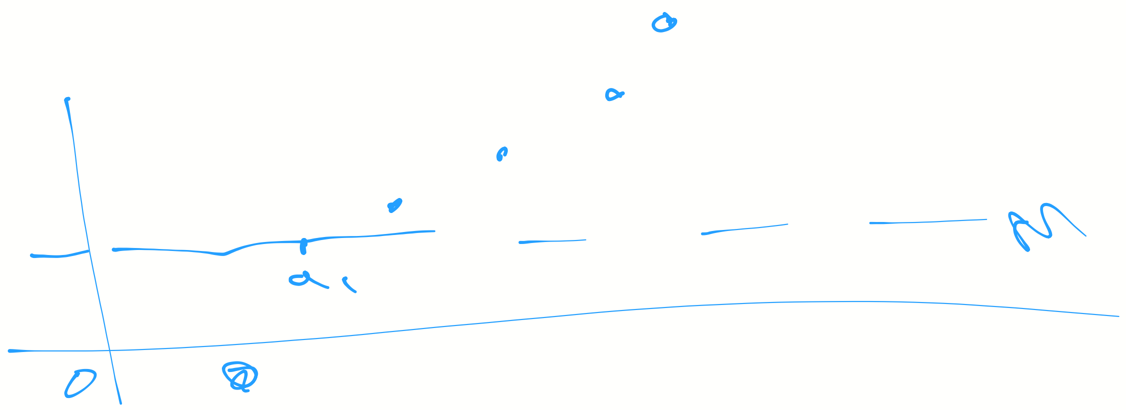
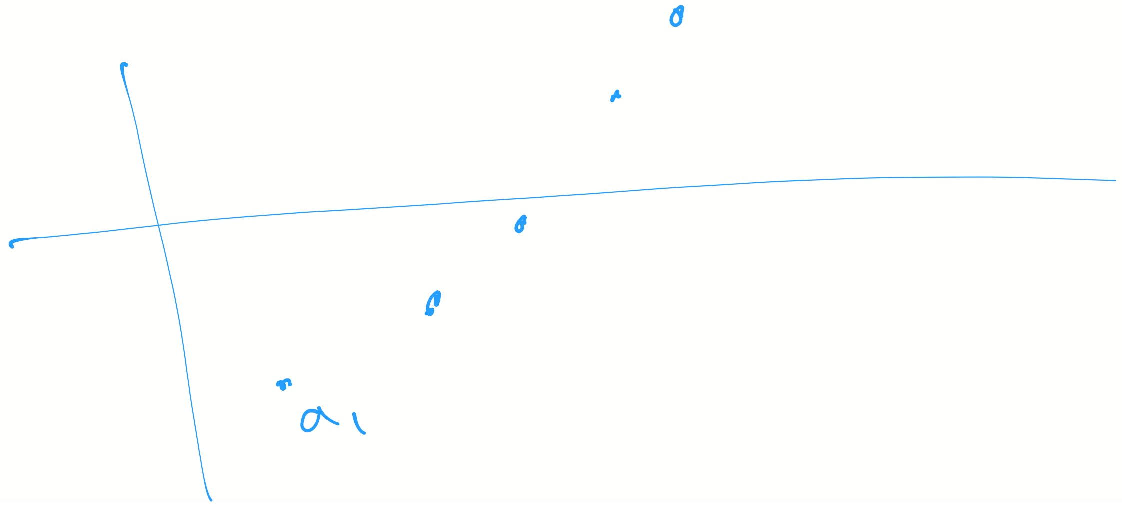
Solutions

3. Prove that a monotone sequence is bounded if it has a bounded subsequence.

Without Loss Of Generality

Solution: **Proof:** WLOG assume that $\{a_n\}$ is a monotone increasing sequence that has a bounded subsequence $\{a_{n_k}\}$, so that there exists a number M with $|a_{n_k}| \leq M$ for all indices k. If k is any index, then $n_k \geq k$ since $\{n_k\}$ is a strictly increasing sequence of natural numbers, so that $a_k \leq a_{n_k}$ by definition of monotone increasing. Note that because this sequence is increasing, it is bounded below by a_1. Thus $|a_k| \leq \max\{|a_1|, |a_{n_k}|\} \leq \max\{|a_1|, M\}$ for all indices k, that is, that $\{a_n\}$ is bounded. ∎





The Extreme Value Theorem (3.2)

*Theme - boundedness

continuity

Extreme Value Theorem

WI -

For a function $f : D \to \mathbb{R}$, we define

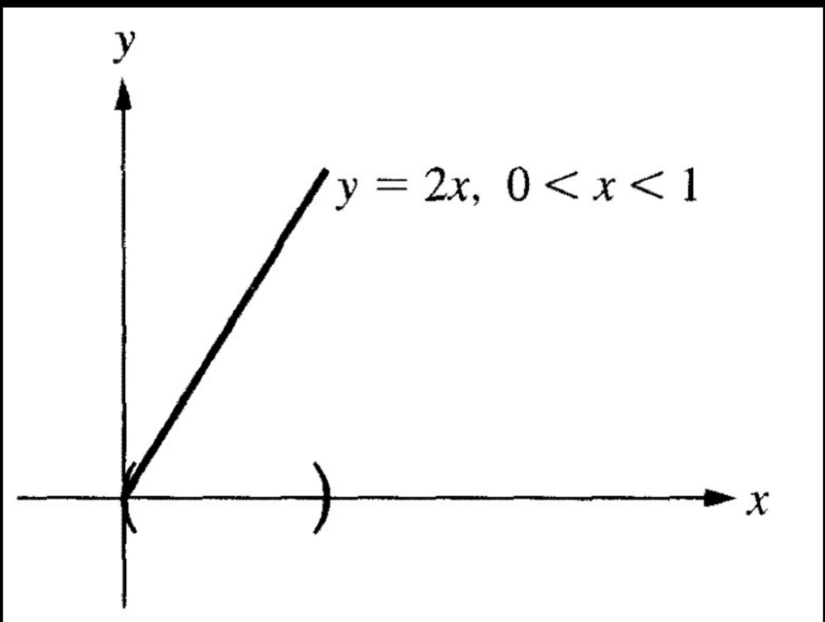
$$f(D) \equiv \{y \mid y = f(x) \qquad \text{for some } x \text{ in } D\}$$

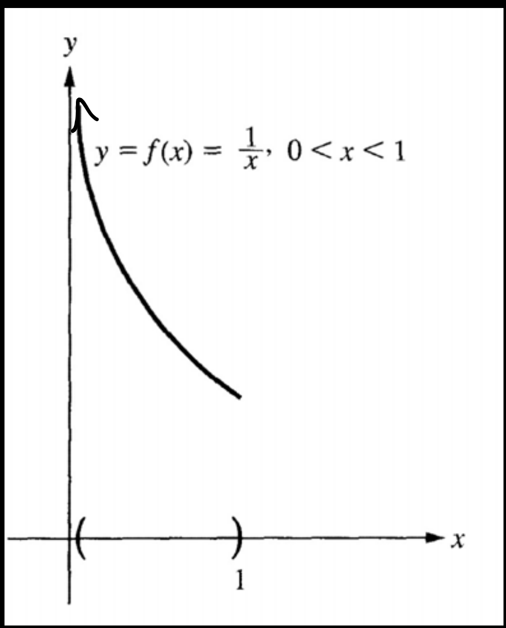
and call the set $f(D)$ the *image* of the function $f : D \to \mathbb{R}$. We say that the function $f : D \to \mathbb{R}$ attains a *maximum value* provided that its image $f(D)$ has a maximum; that is, there is a point x_0 in D such that

$$f(x) \leq f(x_0) \qquad \text{for all } x \text{ in } D.$$

We will call such a point x_0 in D a *maximizer* of the function $f : D \to \mathbb{R}$. Similarly, the function $f : D \to \mathbb{R}$ is said to attain a *minimum value* provided that its image $f(D)$ has a minimum; a point in D at which this minimum value is attained is called a *minimizer* of the function $f : D \to \mathbb{R}$.

?- What about the functions is preventing a max or min from being attained?





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What is the intuition here?

Lemma 3.10 The image of a continuous function on a closed bounded interval,

$$f : [a, b] \to \mathbb{R},$$

is bounded above; that is, there is a number M such that

$$f(x) \leq M \qquad \text{for all } x \text{ in } [a, b].$$

*An important result...

Theorem 3.9 The Extreme Value Theorem A continuous function on a closed bounded interval,

$$f : [a, b] \to \mathbb{R},$$

attains both a minimum and a maximum value.

(EVT)

Theorem 3.9 (Extreme Value Theorem) *A continuous function on a closed bounded interval,*

$$f : [a, b] \to \mathbb{R},$$

attains both a minimum and a maximum value.

Proof. Define

$$S \equiv f([a,b]).$$

Then S is a nonempty set of real numbers that, by the preceding lemma[1], is bounded above. According to the Completeness Axiom, S has a supremum. Define

$$c \equiv \text{supremum}(S).$$

It is necessary to find a point x_0 in $[a,b]$ such that

$$c = f(x_0).$$

Let n be a natural #. Then the number $c - 1/n$ is smaller than c and is therefore not an upper bound for the set S. Thus, there is a point $x \in [a, b]$ such that

$$f(x) > c - 1/n.$$

Choose such a point and label it x_n. From this choice and from the fact that c is an upper bound for S, we see that

$$c - 1/n < f(x_n) \leq c$$

for every index n. Hence the sequence

$$\{f(x_n)\} \text{ converges to } c.$$

The Sequential Compactness Thm. asserts that there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ that converges to a point x_0 in $[a, b]$.
Since $f : [a, b] \to \mathbb{R}$ is continuous at x_0,

$$\{f(x_{n_k})\} \to f(x_0).$$

But $\{f(x_{n_k})\}$ is a subsequence of $\{f(x_n)\}$ that converges to c, so

$$c = f(x_0).$$

The point x_0 is therefore a maximizer of the function → So $f(x_0)$ exists!

[1]**Lemma 3.10.** *The image of a continuous function on a closed bounded interval, $f : [a, b] \to \mathbb{R}$, is bounded above; that is, there is a number M such that $f(x) \leq M$ for all $x \in [a, b]$.*

$$f : [a, b] \to \mathbb{R}.$$

To complete the proof, we observe that the function

$$-f : [a, b] \to \mathbb{R}$$

is also continuous. Consequently, using what we have just proven, we can select a point in $[a, b]$ at which $-f$ attains a maximum value, and at this point the function

$$f : [a, b] \to \mathbb{R}$$

attains a minimum value.

■

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(IVT)

The Intermediate Value Theorem (3.3)

*Theme - We've talked about maxes and mins, but what about functional values "in-between" other values?

WT -

Approximate a solution to the equation $2x^2 - \frac{5}{2}x - \frac{3}{4} = 0$ for $x \in (-1, 2)$, without using the Quadratic Formula.

example strategy:

Note: $2(1)^2 - \frac{5}{2}(1) - \frac{3}{4} = 2 - \frac{5}{2} - \frac{3}{4}$

$= \frac{8}{4} - \frac{10}{4} - \frac{3}{4}$

$= \frac{-5}{4} < 0$

$2(1.5)^2 - \frac{5}{2}(1) - \frac{3}{4} = \ldots > 0$

Theorem 3.11 The Intermediate Value Theorem Suppose that the function $f\colon [a, b] \to \mathbb{R}$ is continuous. Let c be a number strictly between $f(a)$ and $f(b)$; that is,

$f(x_0) = c$ $\qquad\qquad$ $f(x_0) = c$

$$f(a) < c < f(b) \quad \text{or} \quad f(b) < c < f(a).$$

Then there is a point x_0 in the open interval (a, b) at which $f(x_0) = c$.

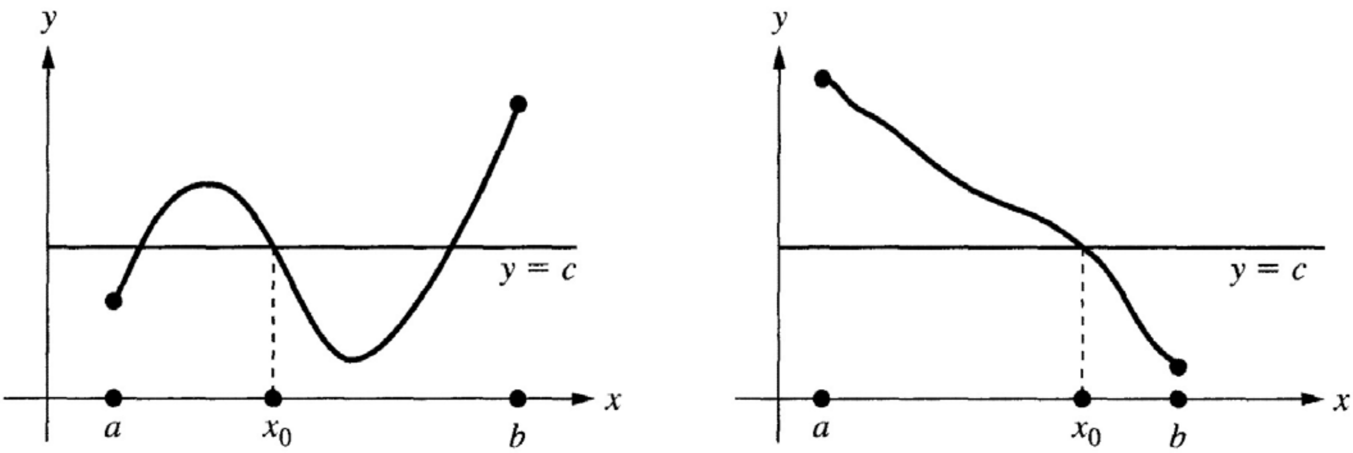


FIGURE 3.4 A number between two functional values is also a functional value.

*Other results . . .

→ not a focus for our class (i.e., not on a quiz, exam)

Definition A subset D of $\mathbb{R}$ is said to be *convex* provided that whenever the points u and v are in D and $u < v$, then the whole interval $[u, v]$ is contained in D.

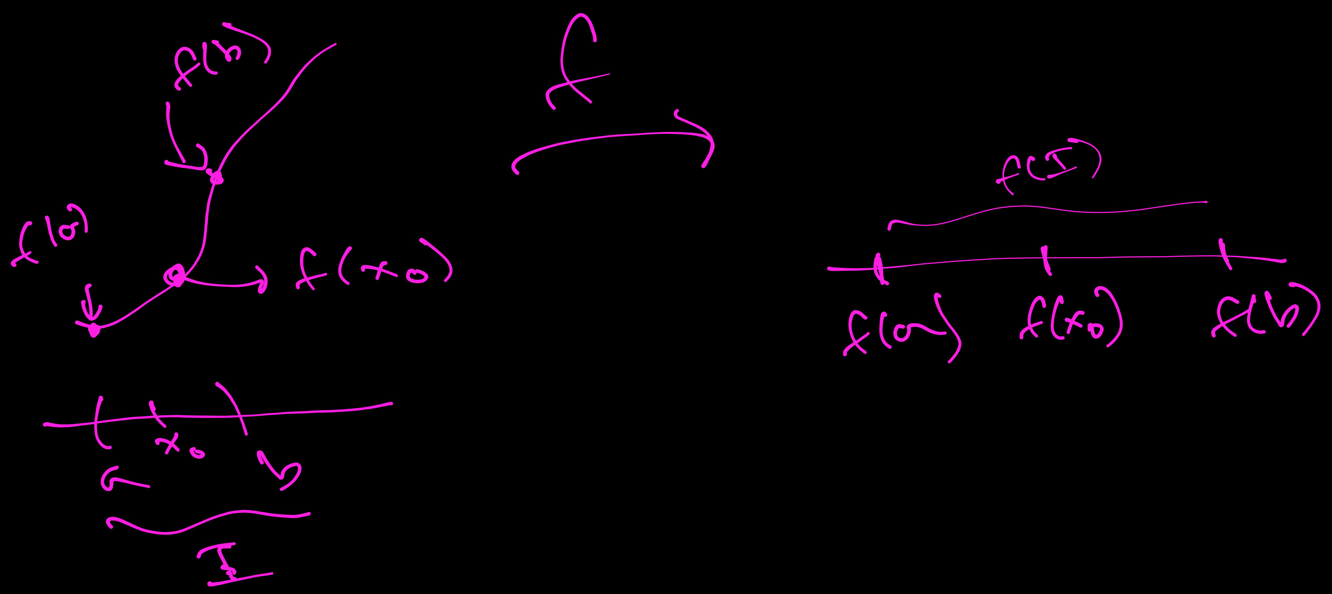
Theorem 3.14 Let I be an interval and suppose that the function $f: I \to \mathbb{R}$ is continuous. Then its image $f(I)$ also is an interval.

Proof * In the text (problem 11 of this section), they prove that convex subsets of $\mathbb{R}$ are intervals.

We show that the image $f(I)$ is a convex set. Let y_1 and y_2 be points in $f(I)$, with $y_1 < y_2$. We must show that the closed interval $[y_1, y_2]$ is also contained in $f(I)$. Indeed, let $y_1 < c < y_2$. Since y_1 and y_2 are in $f(I)$, there are points x_1 and x_2 in I with $f(x_1) = y_1$ and $f(x_2) = y_2$. If we let J be the closed interval having x_1 and x_2 as endpoints, then J is contained in I since, by assumption, the set I is an interval and therefore is convex. Thus, we can apply the Intermediate Value Theorem to the function $f: J \to \mathbb{R}$ in order to conclude that there is a point x_0 in J at which $f(x_0) = c$. Thus, x_0 belongs to I and $f(x_0) = c$. It follows that $[y_1, y_2]$ is contained in $f(I)$. ■

?→ The text calls Theorem 3.14 a "slight generalization" of the IVT.

What is the reasoning behind this characterization of Theorem 3.14?



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Uniform Continuity (3.4)

WP-

Interval Notation

For a pair of real numbers a and b such that $a < b$, we define

$$(a, b) \equiv \{x \text{ in } \mathbb{R} \mid a < x < b\},$$
$$[a, b] \equiv \{x \text{ in } \mathbb{R} \mid a \le x \le b\},$$
$$(a, b] \equiv \{x \text{ in } \mathbb{R} \mid a < x \le b\},$$

and

$$[a, b) \equiv \{x \text{ in } \mathbb{R} \mid a \le x < b\}.$$

Moreover, it is convenient to use the symbols ∞ and $-\infty$ in the following manner. We define

$$[a, \infty) \equiv \{x \text{ in } \mathbb{R} \mid a \le x\},$$
$$(-\infty, b] \equiv \{x \text{ in } \mathbb{R} \mid x \le b\},$$
$$(a, \infty) \equiv \{x \text{ in } \mathbb{R} \mid a < x\},$$
$$(-\infty, b) \equiv \{x \text{ in } \mathbb{R} \mid x < b\},$$

and

$$(-\infty, \infty) \equiv \mathbb{R}.$$

(a) Is $[2,3] \cup [3,4)$ an interval? Yes!

$\hookrightarrow [2,4)$

(b) Is $[2,3) \cup (3,4)$ an interval? No!

(c) Is $\mathbb{Q}$ an interval? No!

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Unif. cont.

Continuity

Definition A function $f : D \to \mathbb{R}$ is said to be *uniformly continuous* provided that whenever $\{u_n\}$ and $\{v_n\}$ are sequences in D such that

$$\lim_{n\to\infty} [u_n - v_n] = 0,$$

then

$$\lim_{n\to\infty} [f(u_n) - f(v_n)] = 0.$$

Note carefully that in the above definition of uniform continuity there is no information regarding convergence of the sequence $\{u_n\}$ to a point in the domain of f. The concept of uniform continuity of f on D is formulated to capture the intuitive notion that "the difference $f(u) - f(v)$ becomes arbitrarily small for any two points u and v in D that are sufficiently close to each other, no matter where the two points are located in the domain."

If a function $f : D \to \mathbb{R}$ is uniformly continuous, then it is continuous.

Theorem 3.17 A continuous function on a closed bounded interval,

$$f : [a, b] \to \mathbb{R},$$

is uniformly continuous.

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The ε-δ Criterion for Continuity (3.5)

≠ Desmos exploration (click me!)

Definition The ϵ-δ Criterion at a Point A function $f : D \to \mathbb{R}$ is said to satisfy the ϵ-δ criterion at a point x_0 in the domain D provided that for each positive number ϵ there is a positive number δ such that for x in D,

$$|f(x) - f(x_0)| < \epsilon \qquad \text{if } |x - x_0| < \delta. \tag{3.13}$$

In terms of the graph of the function $f : D \to \mathbb{R}$, the ϵ-δ criterion at a point x_0 in D can be reworded as follows: *For each symmetric band of width* 2ϵ *about the line* $y = f(x_0)$ *(no matter how small this width is)*, there is an interval $(x_0 - \delta, x_0 + \delta)$, centered at x_0 and of diameter $2\delta > 0$, such that the graph of the restriction of f to this interval lies in the given band.

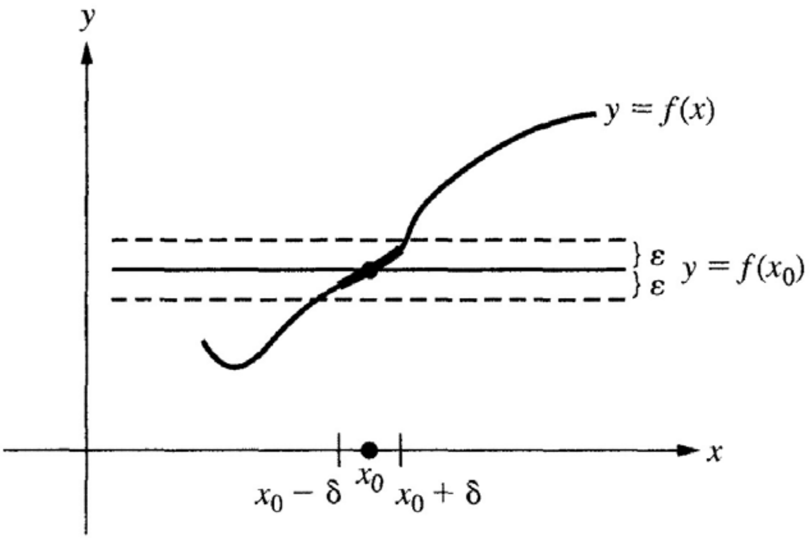

FIGURE 3.6 The ϵ-δ criterion for continuity at x_0.

Theorem 3.20 For a function $f: D \to \mathbb{R}$ and a point x_0 in its domain D, the following two assertions are equivalent:

i. The function $f: D \to \mathbb{R}$ is continuous at x_0; that is, for a sequence $\{x_n\}$ in D,

$$\lim_{n\to\infty} f(x_n) = f(x_0) \qquad \text{if } \lim_{n\to\infty} x_n = x_0.$$

ii. The ϵ-δ criterion at the point x_0 holds; that is, for each positive number ϵ there is a positive number δ such that for x in D,

$$|f(x) - f(x_0)| < \epsilon \qquad \text{if } |x - x_0| < \delta. \tag{3.17}$$

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ex) Prove that $f(x) = x^3$ is continuous at $x = 2$.

Scrap

$$|x - 2| < \delta \longrightarrow |x^3 - 8| < \varepsilon$$

x, x_0, f(x), f(2)

$$|(x-2)(x^2+2x+4)| < \varepsilon$$

$$|x-2||x^2+2x+4| < \varepsilon$$

$$|x-2|[x^2+2|x|+4] < \varepsilon$$

$$|x^2+2x+4| \le |x^2+2x| + |4| \le |x^2| + |2x| + |4|$$

Δ inea.

$$\le x^2 + 2|x| + 4$$

2nd iteration of Δ inea.

Assume $|x-2| < 1$,

So $-1 < x - 2 < 1$

$1 < x < 3$, $\longrightarrow$ * $-3 < x < 3$

$|x| < 3$

and $1 < x^2 < 9$.

Then: $|x-2|[x^2+2|x|+4] < |x-2|[9+2(3)+4]$

*Notice I did not write:

$$|x-2|[x^2+2|x|+4] < (1)[9+2(3)+4]$$
$$< (1)19$$

Now we want $|x-2|[19] < \varepsilon$, or $|x-2| < \dfrac{\varepsilon}{19}$.

So, let's make $\delta = \min\left(1, \frac{\varepsilon}{19}\right)$.

so both of our assumptions are satisfied!

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Proof:

Let $\varepsilon > 0$. Assume $\delta = \min\left(1, \frac{\varepsilon}{19}\right)$. Then observe that by the Triangle Inequality, if $|x-2| < \delta$, we have the following:

$$|x^3 - 8| \leq |x-2|\left[x^2 + 2|x| + 4\right] < |x-2|[19]$$

That is, if $|x-2| < \delta$, then we have the following scenarios:

- If $\delta = 1$ (i.e., $1 < \frac{\varepsilon}{19}$, or $19 < \varepsilon$), then
  $|x-2|[19] < (1)(19) < \varepsilon$.

- If $\delta = \frac{\varepsilon}{19}$ (i.e., $\frac{\varepsilon}{19} < 1$), then
  $|x-2|[19] < \left(\frac{\varepsilon}{19}\right)(19) = \varepsilon$.

Thus, for all $\varepsilon > 0$, we've found $\delta > 0$ such that, if $|x-2| < \delta$, then $|x^3 - 8| < \varepsilon$. That is, $f(x) = x^3$ is continuous at $x = 2$ by definition. $\blacksquare$

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ex)

Example 3.19 Define

$$f(x) = \begin{cases} -x & \text{if } x \le 0 \\ x - 1 & \text{if } x > 0. \end{cases}$$

The function $f : \mathbb{R} \to \mathbb{R}$ does not satisfy the ϵ-δ criterion at the point $x_0 = 0$. Indeed, take $\epsilon \equiv 1/2$. Then there is no positive number δ having the property that

$$-1/2 < f(x) < 1/2 \qquad \text{if } -\delta < x < \delta$$

since no matter what positive number δ is selected, there are positive numbers x in the interval $(-\delta, \delta)$ such that $f(x) < -1/2$.

why is this true?

why is it an issue?

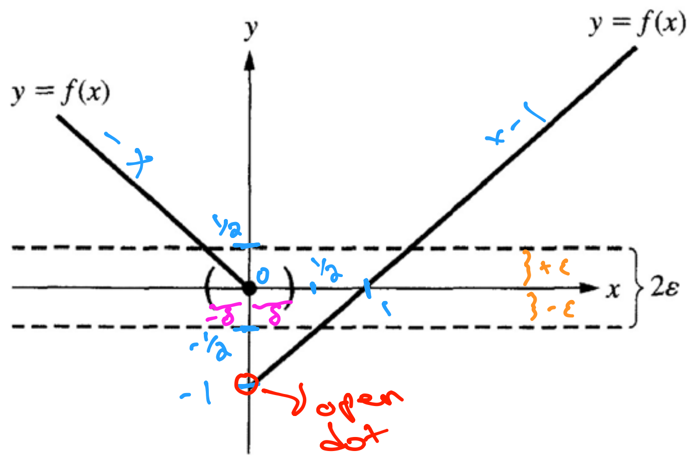

FIGURE 3.7 The ϵ-δ criterion for continuity is not satisfied at the point $x_0 = 0$. ■

Write out the algebra showing the above issue:

SCRAP

Assume that we have continuity at $x=0$.

$|x-0| < \delta \to |f(x) - f(0)| < \varepsilon$

$|x| < \delta \to |f(x) - 0| < \varepsilon$

$|x| < \delta \to |f(x)| < \varepsilon$

$|x| < \delta \to |f(x)| < 1/2$

Suppose $x \in (0, \delta)$, so $f(x) = x - 1$.

$|x| < \delta \to |x-1| < 1/2$

*If $0 < x < 1/2$, then $x - 1 < -1/2$, so $|x-1| > 1/2$.

Proof:

Assume $f(x)$ is continuous at $x=0$. Then for $\varepsilon = 1/2$, by definition of continuity, if $|x|<\delta$, for $\delta>0$, then $|x-1|<1/2$.

Now let $x \in (0, 1/2)$. Observe the following:

$$|x-1| < 1/2$$

$$-1/2 < x-1 < 1/2$$

$$1/2 < x < 3/2$$

* Alternatively: If $0 < x < 1/2$, then $x-1 < -1/2$, so $|x-1| > 1/2$

That is $x > 1/2$, a contradiction!

Hence f is discontinuous at $x=0$. ∎

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Definition The ϵ-δ Criterion on the Domain A function $f: D \to \mathbb{R}$ is said to satisfy the ϵ-δ criterion on the domain D provided that for each positive number ϵ there is a positive number δ such that for all u, v in D,

$$|f(u) - f(v)| < \epsilon \qquad \text{if } |u - v| < \delta. \tag{3.21}$$

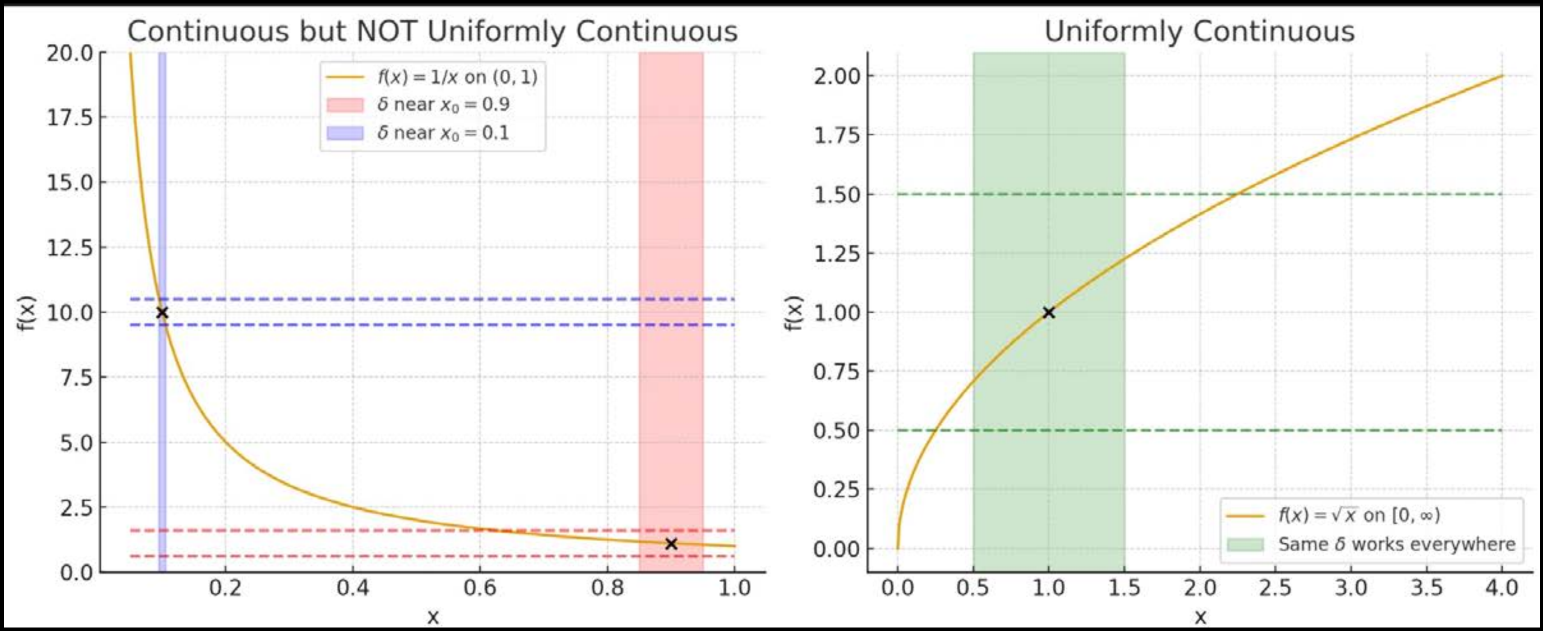
Theorem 3.22 For a function $f: D \to \mathbb{R}$, the following two assertions are equivalent:

i. The function $f: D \to \mathbb{R}$ is uniformly continuous; that is, for two sequences $\{u_n\}$ and $\{v_n\}$ in D,

$$\lim_{n\to\infty} [f(u_n) - f(v_n)] = 0 \qquad \text{if } \lim_{n\to\infty} [u_n - v_n] = 0.$$

ii. The function $f: D \to \mathbb{R}$ satisfies the ϵ-δ criterion at the domain D; that is, for each positive number ϵ there is a positive number δ such that for u, v in D,

$$|f(u) - f(v)| < \epsilon \qquad \text{if } |u - v| < \delta. \tag{3.22}$$


Continuous but NOT Uniformly Continuous
$f(x) = 1/x$ on $(0, 1)$
δ near $x_0 = 0.9$
δ near $x_0 = 0.1$
f(x)
x
Uniformly Continuous
$f(x) = \sqrt{x}$ on $[0, \infty)$
Same δ works everywhere
f(x)
x

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$$\forall x_0 \in S \, \forall \varepsilon > 0 \, \exists \delta > 0 \, \forall x \in S \left[|x - x_0| < \delta \implies |f(x) - f(x_0)| < \varepsilon \right].$$

$$\forall \varepsilon > 0 \, \exists \delta > 0 \, \forall x_0 \in S \, \forall x \in S \left[|x - x_0| < \delta \implies |f(x) - f(x_0)| < \varepsilon \right]$$

The only difference between the two definitions is the order of the quantifiers. When you prove f is continuous your proof will have the form

> *Choose* $x_0 \in S$. *Choose* $\varepsilon > 0$. *Let* $\delta = \delta(x_0, \varepsilon)$. *Choose* $x \in S$. *Assume* $|x - x_0| < \delta$. $\cdots$ *Therefore* $|f(x) - f(x_0)| < \varepsilon$.

The expression for $\delta(x_0, \varepsilon)$ can involve both x_0 and ε but must be independent of x. The order of the quanifiers in the definition signals this; in the proof x has not yet been chosen at the point where δ is defined so the definition of δ must not involve x. (The $\cdots$ represent the proof that $|f(x) - f(x_0)| < \varepsilon$ follows from the earlier steps in the proof.) When you prove f is uniformly continuous your proof will have the form

> *Choose* $\varepsilon > 0$. *Let* $\delta = \delta(\varepsilon)$. *Choose* $x_0 \in S$. *Choose* $x \in S$. *Assume* $|x - x_0| < \delta$. $\cdots$ *Therefore* $|f(x) - f(x_0)| < \varepsilon$.

so the expression for δ can only involve ε and must not involve either x or x_0.

It is obvious that a uniformly continuous function is continuous: if we can find a δ which works for all x_0, we can find one (the same one) which works for any particular x_0. We will see below that there are continuous functions which are not uniformly continuous.

Source: Dr. Joel Robbin (University of Wisconsin-Madison)
(click me!)

Images and Inverses; Monotone Functions (3.6)

Definition The function $f: D \to \mathbb{R}$ is called *monotonically increasing* provided that

$$f(v) \geq f(u) \qquad \text{for all points } u \text{ and } v \text{ in } D \text{ such that } v > u.$$

The function $f: D \to \mathbb{R}$ is called *monotonically decreasing* provided that

$$f(v) \leq f(u) \qquad \text{for all points } u \text{ and } v \text{ in } D \text{ such that } v > u.$$

A function that is either monotonically increasing or monotonically decreasing is said to be *monotone*.

We will now show that a monotone function has the remarkable property that it is continuous if its image is an interval. This result stands in sharp contrast to many results in this chapter in which continuity of the function has been the *assumption*, not the *conclusion*.

e.g., $[2,3)$ or $(4,\infty)$, but <u>not</u> $[2,3) \cup (4,\infty)$

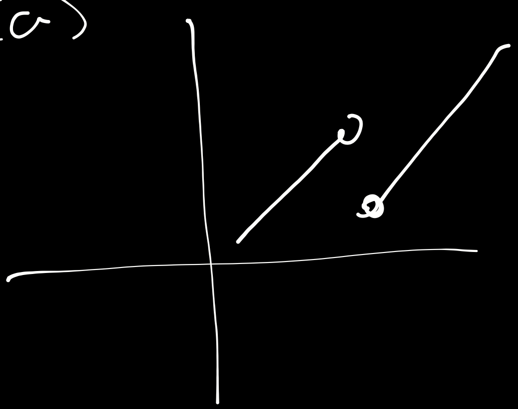
Theorem 3.23 Suppose that the function $f: D \to \mathbb{R}$ is monotone. If its image $f(D)$ is an interval, then the function f is continuous.

Even Stronger!

Corollary 3.25 Let I be an interval and suppose that the function $f: I \to \mathbb{R}$ is monotone. Then the function f is continuous if and only if its image $f(I)$ is an interval.

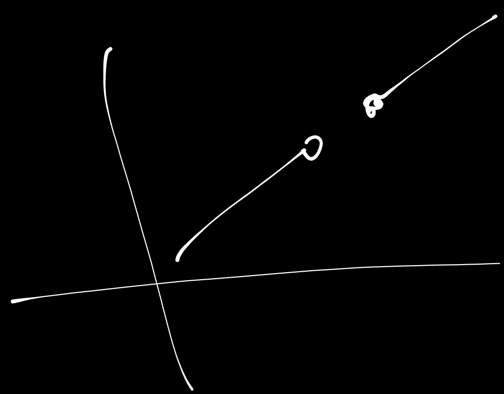
Counterexamples of changes to Thm. 3.23

(a)



Not monotone

(b)



Monotone, but $f(D)$ is <u>not</u> an interval

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* How about "strict" monotonicity?

Definition The function $f: D \to \mathbb{R}$ is called *strictly increasing* provided that

$$f(v) > f(u) \qquad \text{for all points } u \text{ and } v \text{ in } D \text{ such that } v > u.$$

The function $f: D \to \mathbb{R}$ is called *strictly decreasing* provided that

$$f(v) < f(u) \qquad \text{for all points } u \text{ and } v \text{ in } D \text{ such that } v > u.$$

A function that is either strictly increasing or strictly decreasing is said to be *strictly monotone*.

Strictly monotone functions $f: D \to \mathbb{R}$ have the property that for a point y in the image $f(D)$ there is *exactly one point* x in its domain D such that $f(x) = y$. This property is important in general and therefore it has a name.

Definition A function $f: D \to \mathbb{R}$ is said to be *one-to-one* provided that for each point y in its image $f(D)$, there is exactly one point x in its domain D such that $f(x) = y$.

* What about continuity of inverses of functions?

For a function $f: D \to \mathbb{R}$ that is one-to-one, by definition, if y is a point in $f(D)$, there is exactly one point x in D such that $f(x) = y$. We will denote this point x by $f^{-1}(y)$, so we have defined the function

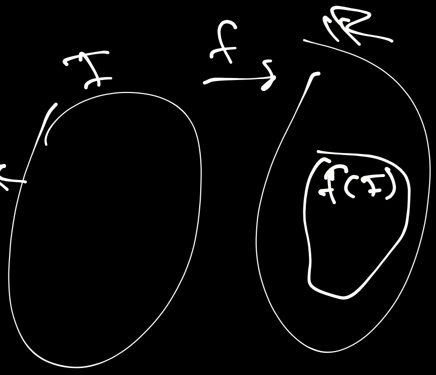
$$f^{-1}: f(D) \to \mathbb{R}, \quad \neq \frac{1}{f}$$

$$f^{-1}(f(x)) = x \qquad \text{for all } x \text{ in } D;$$
$$f(f^{-1}(y)) = y \qquad \text{for all } y \text{ in } f(D).$$

As noted above, strictly monotone functions are one-to-one and therefore have inverse functions.

Theorem 3.29 Let I be an interval and suppose that the function $f : I \to \mathbb{R}$ is strictly monotone. Then the inverse function $f^{-1} : f(I) \to \mathbb{R}$ is continuous.

- f is strictly monotone, so certainly monotone
- Thus, f^{-1} is monotone.
- Note that f^{-1} has an image of f, that is, I
- So, since I is an interval, by Corollary 3.27, $f^{-1}: f(I) \to \mathbb{R}$ is continuous.



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#Other results...

Definition For $x > 0$ and rational number $r = m/n$, where m and n are integers with n positive, we define

$$x^r \equiv (x^m)^{1/n}.$$

Proposition 3.30 For r a rational number, define

$$f(x) = x^r \qquad \text{for } x \geq 0.$$

The function $f : [0, \infty) \to \mathbb{R}$ is continuous.

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Limits (3.7)

i.e., $x_n \in D$ for all n, but $x_n \neq x_0$.

Definition For a set D of real numbers, the number x_0 is called a *limit point of* $\mathbb{D}$ provided that there is a sequence of points in $D\backslash\{x_0\}$ that converges to x_0.

Definition Given a function $f: D \to \mathbb{R}$ and a limit point x_0 of its domain D, for a number ℓ, we write

$$\lim_{x \to x_0} f(x) = \ell \tag{3.30}$$

provided that whenever $\{x_n\}$ is a sequence in $D\backslash\{x_0\}$ that converges to x_0,

$$\lim_{n \to \infty} f(x_n) = \ell.$$

We read (3.30) as "The limit of $f(x)$ as x approaches x_0, with x in D, equals ℓ."

Theorem 3.37 For functions $f: D \to \mathbb{R}$ and $g: U \to \mathbb{R}$, suppose that x_0 is a limit point of D such that

$$\lim_{x \to x_0} f(x) = y_0 \tag{3.39}$$

and that y_0 is a limit point of U such that

$$\lim_{y \to y_0} g(y) = \ell. \tag{3.40}$$

Moreover, suppose that

$$f(D\backslash\{x_0\}) \text{ is contained in } U\backslash\{y_0\}. \tag{3.41}$$

Then

$$\lim_{x \to x_0} (g \circ f)(x) = \ell.$$

WT-

Find a continuous function $f:\mathbb{R}\to\mathbb{R}$ such that when showing f is continuous at $c\in\mathbb{R}$, we find that δ is not dependent on ε.

ex) Let $f(x)=c$.

WTS: $|x-x_0|<\delta \to |f(x)-f(x_0)|<\varepsilon$

$|c-c|=0$

Proof: Let $\varepsilon>0$. Observe that for all δ, $|f(x)-f(x_0)|=|c-c|=0<\varepsilon$. Thus, if $|x-x_0|<\delta$, then $|f(x)-f(x_0)|<\varepsilon$, but δ is not dependent on ε! ∎

The Algebra of Derivatives (4.1)

* Theme: What's happening on the "edge" (i.e., think tangent lines!) of a function?

Tangent Lines and Derivatives

To make the above precise, we need to define the *tangent line*. For a function $f : I \to \mathbb{R}$, where I is a neighborhood of the point x_0, observe that for a point x in I, with $x \neq x_0$, the slope of the line joining the points $(x_0, f(x_0))$ and $(x, f(x))$ is

$$\frac{f(x) - f(x_0)}{x - x_0}.$$

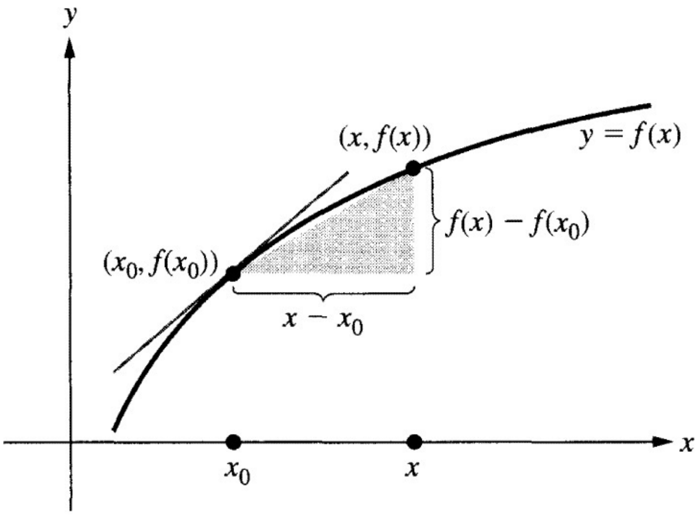

FIGURE 4.1 Approximation of the slope of the tangent line at the point $(x_0, f(x_0))$.

It is reasonable to expect that if there is a tangent line to the graph of $f : I \to \mathbb{R}$ at $(x_0, f(x_0))$, which has a slope m_0, then one should have

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = m_0.$$

For a number x_0, an open interval $I = (a, b)$ that contains x_0 is called a *neighborhood* of x_0.

$x_0 \in (a, b)$

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Definition Let I be a neighborhood of x_0. Then the function $f : I \to \mathbb{R}$ is said to be differentiable at x_0 provided that

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \tag{4.2}$$

exists, in which case we denote this limit by $f'(x_0)$ and call it the derivative of f at x_0; that is,

"is defined as..."

$$f'(x_0) \equiv \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}. \tag{4.3}$$

If the function $f : I \to \mathbb{R}$ is differentiable at every point in I, we say that f is *differentiable* and call the function $f' : I \to \mathbb{R}$ the *derivative* of f.

* See text for review and proof of derivative rules!

Differentiable Functions are Continuous

Proposition 4.5 Let I be a neighborhood of x_0 and suppose that the function $f : I \to \mathbb{R}$ is differentiable at x_0. Then f is continuous at x_0.

Proof

Since

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0) \qquad \text{and} \qquad \lim_{x \to x_0} [x - x_0] = 0,$$

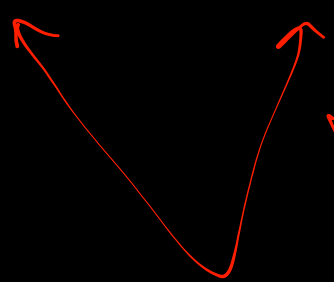
it follows from the product property of limits that

$$\lim_{x \to x_0} [f(x) - f(x_0)] = \lim_{x \to x_0} \left[\frac{f(x) - f(x_0)}{x - x_0} \cdot (x - x_0) \right] = f'(x_0) \cdot 0 = 0.$$

Thus, $\lim_{x \to x_0} f(x) = f(x_0)$, which means that f is continuous at x_0. ■

As Example 4.3 shows, it is not true that continuity of a function at a point implies the differentiability of the function at that point.

*



$|x|$

✓ Continuous

✗ Differentiable at $x=0$

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Leibnitz Notation for derivative

Use induction to prove the differentiation formula $\frac{d}{dx}(x^n) = nx^{n-1}$ for all natural numbers n, making use of the differentiation formulas $\frac{d}{dx}(x) = 1$ and $\frac{d}{dx}(fg) = f\frac{dg}{dx} + g\frac{df}{dx}$, that is, the product rule.

First, note that for $n=1$, $\frac{d}{dx}x^n = \frac{d}{dx}x^1 = 1$ (by the given differentiation formula). That is, since $1 = 1x^{1-1} = 1$, we have that $\frac{d}{dx}x^n = nx^{n-1}$ for $n=1$.

Assume $\frac{d}{dx}x^n = nx^{n-1}$ for some $n \in \mathbb{N}$.

Then observe that
$$\frac{d}{dx}x^{n+1} = \frac{d}{dx}[x^n \cdot x^1]$$
$$= \frac{d}{dx}x^n \cdot x^1 + x^n \cdot \frac{d}{dx}x^1$$
(by the Product Rule)
$$= (nx^{n-1})x^1 + x^n(1)$$
$$= nx^n \cdot x^{-1} x^1 + x^n$$
$$= nx^n + x^n$$
$$= x^n(n+1)$$
$$= \boxed{(n+1)x^n}$$

By the Principle of Mathematical Induction, $\frac{d}{dx}x^n = nx^{n-1}$ for all $n \in \mathbb{N}$. ∎

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Differentiating Inverses and Compositions (4.2)

#1 (a^*) → return to previous class's WP!

Theorem 3.29 asserts that if I is an interval and the function $f : I \to \mathbb{R}$ is strictly monotone with image $J = f(I)$, then its inverse function $f^{-1} : J \to \mathbb{R}$ is continuous. It is natural to consider the question of the differentiability of the inverse function f^{-1} at the point $y_0 \equiv f(x_0)$ in J if f is differentiable at x_0. We will show that if f is differentiable at x_0 with $m \equiv f'(x_0)$, then if $m \neq 0$, the inverse function f^{-1} is differentiable at y_0 and its derivative at y_0 equals $1/m$. Before proving this, we explain geometrically why this formula is natural.

Indeed, suppose that the function $f : I \to \mathbb{R}$ is differentiable at x_0 and its tangent line ℓ at the point $p = (x_0, y_0)$ is not horizontal. This means that

$$m \equiv f'(x_0) \neq 0.$$

Then it appears that the tangent line to the inverse function at the point p is also the same line ℓ. From the viewpoint of the inverse function, the line ℓ is now defined as the graph of a function with domain on the vertical axis, and therefore its slope is the reciprocal of $m \equiv f'(x_0)$.

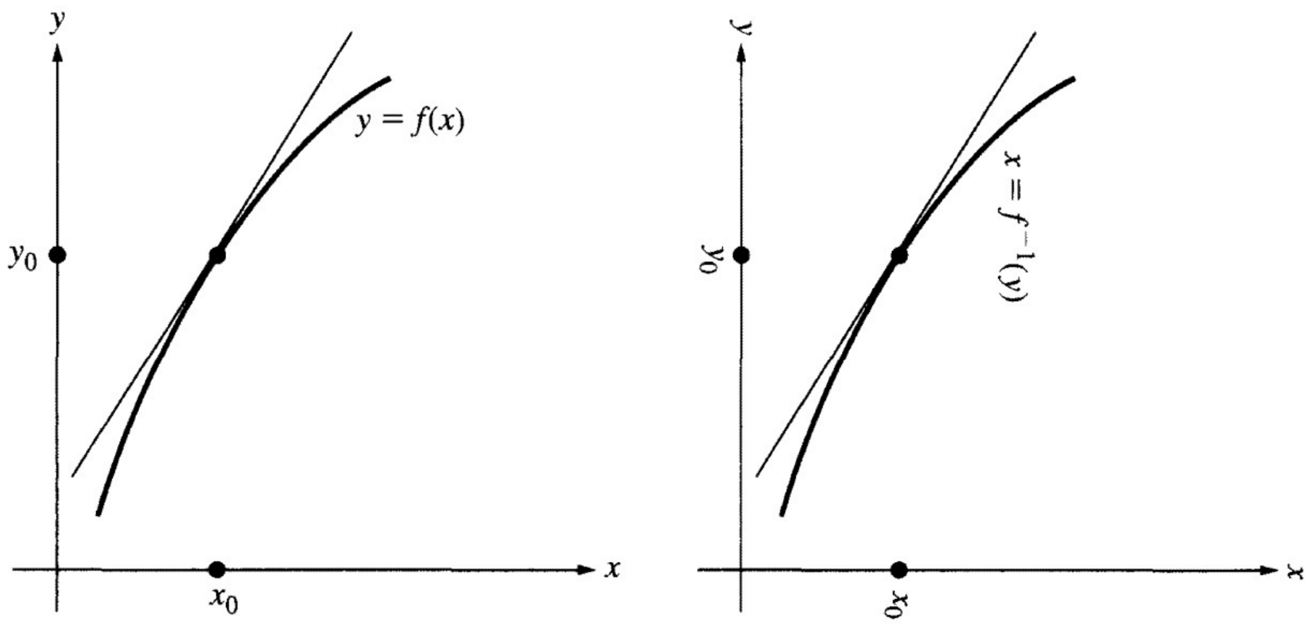

FIGURE 4.3 $(f^{-1})'(y_0) = 1/f'(x_0)$.

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Theorem 4.11 *Let I be a neighborhood of x_0 and let the function $f : I \to \mathbb{R}$ be strictly monotone and continuous. Suppose that f is differentiable at x_0 and that $f'(x_0) \neq 0$. Define $J = f(I)$. Then the inverse $f^{-1} : J \to \mathbb{R}$ is differentiable at the point $y_0 = f(x_0)$ and*

$$(f^{-1})'(y_0) = \frac{1}{f'(x_0)}$$

$I \xrightarrow{f} Q \to f(I) =$

Proof. It follows from the Intermediate Value Theorem that J is a neighborhood of $\underline{y_0} = f(x_0)$. For a point y in J, with $y \neq y_0$, define

$$x \equiv \underline{f^{-1}(y)},$$

so that

$$\frac{f^{-1}(y) - f^{-1}(y_0)}{y - y_0} = \frac{1}{\dfrac{f(x) - f(x_0)}{x - x_0}}.$$

$\dfrac{x - x_0}{f(x) - f(x_0)}$

Since the inverse function $f^{-1} : J \to \mathbb{R}$ is continuous,

$$\lim_{y \to y_0} x = \lim_{y \to y_0} \underline{f^{-1}(y)} = \underline{f^{-1}(y_0)} \equiv x_0.$$

By the composition property for limits, the quotient property of limits, and the definition of the differentiability of $f : I \to \mathbb{R}$ at x_0, it follows that

$$(f^{-1})'(y_0) = \lim_{y \to y_0} \frac{f^{-1}(y) - f^{-1}(y_0)}{y - y_0} = \lim_{y \to y_0} \frac{1}{\dfrac{f(x) - f(x_0)}{x - x_0}} = \frac{1}{f'(x_0)}$$

Thus, $f^{-1} : J \to \mathbb{R}$ is differentiable at $\underline{y_0}$. ■

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*Other important results...

→ More general

Corollary 4.12 Let I be an open interval and suppose that the function $f : I \to \mathbb{R}$ is strictly monotone and differentiable with a nonzero derivative at each point in I. Define $J = f(I)$. Then the inverse function $f^{-1} : J \to \mathbb{R}$ is differentiable and

$$(f^{-1})'(x) = \frac{1}{f'\big(f^{-1}(x)\big)} \qquad \text{for all } x \text{ in } J. \tag{4.6}$$

The Derivative of the Composition

We have shown that the composition of continuous functions is continuous. The composition of differentiable functions is differentiable, and there is a formula for the derivative of the composition. This is the content of the following theorem.

Theorem 4.14 The Chain Rule Let I be a neighborhood of x_0 and suppose that the function $f : I \to \mathbb{R}$ is differentiable at x_0. Let J be an open interval such that $f(I) \subseteq J$ and suppose that the function $g : J \to \mathbb{R}$ is differentiable at $f(x_0)$. Then the composition $g \circ f : I \to \mathbb{R}$ is differentiable at x_0 and

$$(g \circ f)'(x_0) = g'(f(x_0))f'(x_0). \tag{4.7}$$

$\to [g(f(x))]'$

WT:

(a)

Suppose the function $f : \mathbb{R} \to \mathbb{R}$ has the following property:

$$\lim_{x\to 0} \frac{f(x) - f(0)}{x} = \ell$$

.

Prove that, for any natural number k,

$h(x) = x^k$, $\lim_{x\to 0} h(x) = 0$

$$\lim_{x\to 0} \frac{f(x^k) - f(0)}{x^k} = \ell$$

Proof: Note that x^k is a polynomial, and so is continuous. Thus, $\lim_{x\to 0} x^k = (0)^k = 0$. Also note that $\lim_{x\to 0} \frac{f(x) - f(0)}{x} = \ell$. Thus, by the composition property of limits, $\lim_{x\to 0} \frac{f(x^k) - f(0)}{x^k} = \ell$. ∎

(b)

Recall that a function $f : \mathbb{R} \to \mathbb{R}$ is called "even" if $f(-x) = f(x)$ for all x, and f is called "odd" if $f(-x) = -f(x)$ for all x.

$\hookrightarrow f(-x) + f(x)$

Suppose f is odd and continuous.

Show that if $g(x) = \overbrace{f(x) + f}^{=}(-x)$, then we've found another example in which we can prove g is continuous, and that δ is not dependent on ϵ in our proof.

Can you come up with another example?

SCRAP

$|x - x_0| < \delta \longrightarrow |g(x) - g(x_0)| < \epsilon$

$f(x) + f(-x)$ $\quad$ $[f(x_0) + f(-x_0)]$

ex) $f(-x) = f(x)$

$g(x) = f(-x) - f(x)$

$|0 - 0| < \epsilon$

The Mean Value Theorem and Its Geometric Consequences (4.3)

We will now prove one of the most useful and geometrically attractive results in calculus, the Mean Value Theorem[2]. It asserts that if the function $f : [a, b] \to \mathbb{R}$ is continuous and its restriction to the open interval (a, b) is differentiable, then there is a point x_0 in the open interval (a, b) with the property that the tangent line to the graph at the point $(x_0, f(x_0))$ is parallel to the line passing through the points $(a, f(a))$ and $(b, f(b))$.

[2] This theorem is often called the Lagrange Mean Value Theorem in order to distinguish it from the Cauchy Mean Value Theorem that we will prove in the next section.

To prove the Mean Value Theorem, it is convenient first to prove some preliminary results.

Lemma 4.16 Let I be a neighborhood of x_0 and suppose that the function $f : I \to \mathbb{R}$ is differentiable at x_0. If the point x_0 is either a maximizer or a minimizer of the function $f : I \to \mathbb{R}$, then $f'(x_0) = 0$.

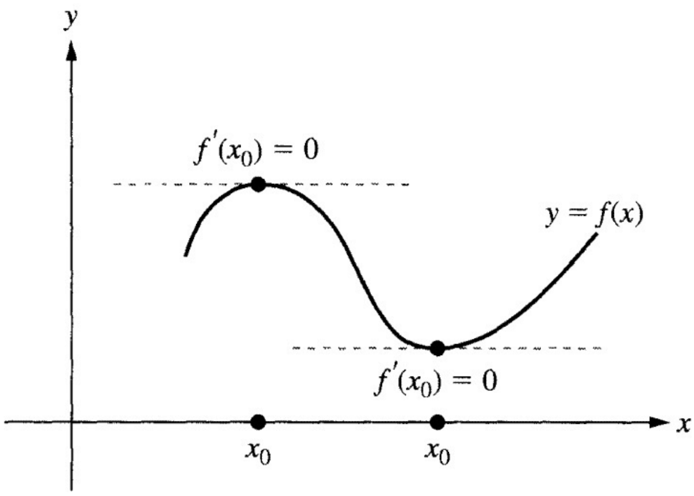

FIGURE 4.4 $f'(x_0) = 0$ if the point x_0 is a minimizer or a maximizer for the function f.

Theorem 4.17 (Rolle's Theorem) *Suppose that the function $f : [a, b] \to \mathbb{R}$ is continuous and that the restriction of f to the open interval (a, b) is differentiable. Assume, moreover, that*

$$\underline{f(a)} = \underline{f(b)}.$$

Then there is a point x_0 in the open interval (a, b) at which

$$\underline{f'(x_0)} = 0. = \frac{\overbrace{f(b)-f(a)}^{=0}}{b-a}$$

see MVT (more general Rolle's thm)

Proof. Since $f : [a, b] \to \mathbb{R}$ is continuous, according to the Extreme Value Theorem, it attains both a max value and a min value on $[a, b]$.
Since $f(a)$ = $f(b)$, if both the maximizers and the minimizers occur at the endpoints, then the function $f : [a, b] \to \mathbb{R}$ is constant, so $f'(x) = 0$ at every point x in (a, b).
Otherwise, the function has either a minimizer or a maximizer at some point x_0 in the open interval $I \equiv (a, b)$, and hence, by the preceding lemma, at this point $f'(x_0) = 0$.
■

Theorem 4.18 (The Mean Value Theorem) *Suppose that the function $f : [a, b] \to \mathbb{R}$ is continuous and that the restriction of f to the open interval (a, b) is differentiable. Then there is a point x_0 in the open interval (a, b) at which*

$$f'(x_0) = \frac{f(b) - f(a)}{b - a}.$$

"auxiliary function"

Proof. For a number m, we wish to apply Rolle's Thm to the function $h : [a, b] \to \mathbb{R}$ defined by

$$h(x) = f(x) - mx$$

for x in $[a, b]$. To do so, we must have $h(a) = h(b)$, and this occurs precisely when we choose

$$m \equiv \frac{f(b) - f(a)}{b - a}.$$

For this choice of m, we apply Rolle's Theorem to choose a point x_0 in the open interval (a, b) at which $h'(x_0) = 0$. Since $h'(x) = f'(x) - m$ at this point x_0,

$$f'(x_0) = \frac{f(b) - f(a)}{b - a}.$$

■

As a general principle, if we have information about the derivative of a function that we wish to use in order to analyze the function, we should first try to apply the Mean Value Theorem. The remainder of this section comprises various applications of this strategy.

$$f'(x_0) = \frac{f(b) - f(a)}{b - a}.$$

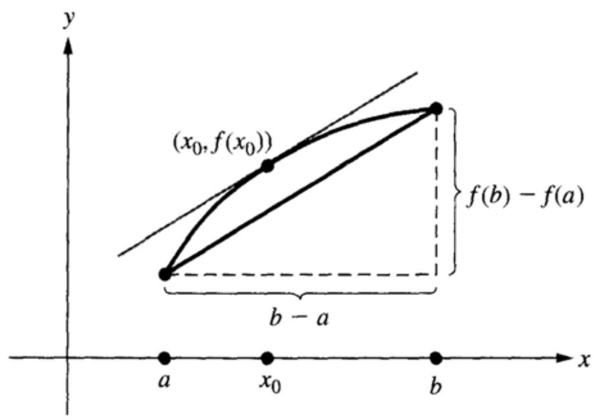

FIGURE 4.5 The tangent line is parallel to the segment joining the endpoints.

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*Other important results...

The Identity Criterion

A function $f : D \to \mathbb{R}$ is said to be *constant* provided that there is some number c such that $f(x) = c$ for all x in D.

Lemma 4.19 Let I be an open interval and suppose that the function $f : I \to \mathbb{R}$ is differentiable. Then $f : I \to \mathbb{R}$ is constant if and only if

$$f'(x) = 0 \qquad \text{for all } x \text{ in } I.$$

Proposition 4.20 The Identity Criterion Let I be an open interval and let the functions $g : I \to \mathbb{R}$ and $h : I \to \mathbb{R}$ be differentiable. Then these functions differ by a constant if and only if

$$g'(x) = h'(x) \qquad \text{for all } x \text{ in } I. \tag{4.10}$$

In particular, these functions are identically equal if and only if (4.10) holds and there is some point x_0 in I at which

$$g(x_0) = h(x_0).$$

A Criterion for Strict Monotonicity

Corollary 4.21 Let I be an open interval and the function $f : I \to \mathbb{R}$ be differentiable. Suppose that $f'(x) > 0$ for all x in I. Then $f : I \to \mathbb{R}$ is strictly increasing.

A Criterion for Selecting Maximizers and Minimizers

Definition A point x_0 in the domain of a function $f : D \to \mathbb{R}$ is said to be a *local maximizer* for f provided that there is some $\delta > 0$ such that

$-\delta < x - x_0 < \delta$

$-\delta + x_0 < x < \delta + x_0$

$$f(x) \leq f(x_0) \qquad \text{for all } x \text{ in } D \text{ such that } |x - x_0| < \delta.$$

We call x_0 a *local minimizer* for f provided that there is some $\delta > 0$ such that

$$f(x) \geq f(x_0) \qquad \text{for all } x \text{ in } D \text{ such that } |x - x_0| < \delta.$$

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For a differentiable function $f : I \to \mathbb{R}$ that has as its domain an open interval I, we say that $f : I \to \mathbb{R}$ has *one derivative* if $f : I \to \mathbb{R}$ is differentiable and define $f^{(1)}(x) = f'(x)$ for all x in I. If the function $f' : I \to \mathbb{R}$ itself has a derivative, we say that $f : I \to \mathbb{R}$ has *two derivatives*, or has a *second derivative*, and denote the derivative of $f' : I \to \mathbb{R}$ by $f'' : I \to \mathbb{R}$ or by $f^{(2)} : I \to \mathbb{R}$. Now let k be a natural number for which we have defined what it means for $f : I \to \mathbb{R}$ to *have k derivatives* and have defined $f^{(k)} : I \to \mathbb{R}$. Then $f : I \to \mathbb{R}$ is said to *have $k + 1$ derivatives* if $f^{(k)} : I \to \mathbb{R}$ is differentiable, and we define $f^{(k+1)} : I \to \mathbb{R}$ to be the derivative of $f^{(k)} : I \to \mathbb{R}$. In this context, is it useful to denote $f(x)$ by $f^{(0)}(x)$.

Theorem 4.22 Let I be an open interval containing the point x_0 and suppose that the function $f : I \to \mathbb{R}$ has a second derivative. Suppose that

$$f'(x_0) = 0.$$

If $f''(x_0) > 0$, then x_0 is a local minimizer of f.

If $f''(x_0) < 0$, then x_0 is a local maximizer of f.

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The Cauchy Mean Value Theorem (4.4)

Theorem 4.23 (The Cauchy Mean Value Theorem) *Suppose that the functions* $f : [a, b] \to \mathbb{R}$ *and* $g : [a, b] \to \mathbb{R}$ *are continuous and that their restrictions to the open interval* (a, b) *are differentiable. Moreover, assume that*

$$\underline{g'(x)} \neq 0 \quad \text{for all } x \in (a, b).$$

Then there is a point x_0 *in the open interval* (a, b) *at which*

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(\underline{x_0})}{g'(\underline{x_0})}.$$

Proof.
To prove this theorem we will use Rolle's Theorem in a manner similar to the way we proved the Mean Value Theorem. For a number m, we wish to apply Rolle's Theorem to the function $h : [a, b] \to \mathbb{R}$ defined by

$$h(x) \equiv f(x) - mg(x) \quad \text{for } x \in [a, b].$$

To do so, we must have $h(a) = h(b)$, and this occurs precisely when we choose

$$m = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

For this choice of m, we apply Rolle's Theorem to choose a point x_0 in the open interval (a, b) at which $\underline{h'(x_0)} = 0$. Since $h'(x) = \underline{f'(x)} - m\underline{g'(x)}$, at this point x_0,

$$\frac{f'(x_0)}{g'(x_0)} = m = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

*Observe that if $g(x) = x$ for $a \leq x \leq b$, then the CMVT reduces to the MVT! ■

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*Used in chapter 8!

Theorem 4.24 *Let I be an open interval and n be a natural number and suppose that the function $f : I \to \mathbb{R}$ has n derivatives. Suppose also that at the point x_0 in I,*

Including the "zero-th" derivative of f(x) itself.

$$f^{(k)}(x_0) = 0 \quad \text{for } 0 \le k \le n-1.$$

Then, for each point $x \neq x_0$ in I, there is a point z strictly between x and x_0 at which

$$f(x) = \frac{f^{(n)}(z)}{n!}(x - x_0)^n.$$

$n \cdot n-1 \cdot n-2 \cdots 1$ *0! = 1

Proof. auxiliary function

Define $g(x) = (x - x_0)^n$ for all x in I. Then

$$g^{(k)}(x_0) = \underline{\quad 0 \quad}, \quad \text{for } 0 \le k \le n-1$$

and

$$g^{(n)}(x_0) = \underline{\quad n! \quad}.$$

Let x be a point in I with $x \neq x_0$. Without loss of generality, suppose that $x > x_0$. Applying the Cauchy Mean Value Theorem to $f : [x_0, x] \to \mathbb{R}$ and $g : [x_0, x] \to \mathbb{R}$, we can select a point x_1 in (x_0, x) at which

$$\frac{f(x)}{g(x)} = \frac{f(x) - \overbrace{f(x_0)}^{=0}}{g(x) - \underbrace{g(x_0)}_{=0}} = \frac{f'(x_1)}{g'(x_1)}.$$

Next, we apply the Cauchy Mean Value Theorem to f' and g' on $[x_0, x_1]$ to select a point x_2 in (x_0, x_1) satisfying

$$\frac{f'(x_1)}{g'(x_1)} = \frac{f'(x_1) - f'(x_0)}{g'(x_1) - g'(x_0)} = \frac{f''(x_2)}{g''(x_2)}.$$

By repeating this process with successively higher derivatives, we eventually obtain a point $\underline{\quad x_n \quad}$ in (x_0, x) such that

$$\frac{f(x)}{g(x)} = \frac{f^{(n)}(x_n)}{g^{(n)}(x_n)} = \frac{f^{(n)}(x_n)}{n!}.$$

This gives

$$f(x) = \frac{f^{(n)}(x_n)}{n!}(x - x_0)^n.$$

Setting $z = x_n$, we obtain the desired result. ■

$g(x) = (x - x_0)^n$

$g'(x) = n(x - x_0)^{n-1}$

$g''(x) = n(n-1)(x - x_0)^{n-2}$

$\vdots$

$g^{(k)}(x) = n(n-1)\cdots(n-(k-1))(x - x_0)^{n-k}$

So $g^{(k)}(x_0) = n(n-1)\cdots(n-(k-1))(x_0 - x_0)^{n-k} = 0$

$g^{(n)}(x) = n(n-1)\cdots(n-(n-1))(x - x_0)^{n-n}$

$= n(n-1)\cdots(1)(x - x_0)^0$

$= n!$

So $g^{(n)}(x_0) = n!$

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Darboux Sums: Upper and Lower Integrals (6.1)

WU:

True or False:

1) $\sum_{i=1}^{n} c a_i = c \sum_{i=1}^{n} a_i$ True

constant

summation ("sigma")

2) $\sum_{i=1}^{n} (a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$ True

3) $\sum_{i=1}^{n} a_i b_i = \sum_{i=1}^{n} a_i \sum_{i=1}^{n} b_i$ False!

Counterexample:

$$\sum_{i=1}^{3} 3 \cdot 2 \stackrel{?}{=} \sum_{i=1}^{3} 3 \cdot \sum_{i=1}^{3} 2$$

$3 \cdot 2 + 3 \cdot 2 + 3 \cdot 2$

$=$

$(3+3+3)(2+2+2)$

$=$

For certain functions $f : [a, b] \to \mathbb{R}$ that we call *integrable*, we will define a number called the *integral* of f on $[a, b]$ and denoted by $\int_a^b f$. Four principal goals of this chapter are to

i. Define the concepts of integrable function and integral and then establish a criterion for integrability we call the Archimedes–Riemann Theorem.
ii. Prove that a continuous function $f : [a, b] \to \mathbb{R}$ is integrable.
iii. Prove the First Fundamental Theorem (Integrating Derivatives), which states that for a continuous function $F : [a, b] \to \mathbb{R}$ that has a continuous bounded derivative on the open interval (a, b), the following integration formula holds:

$$\int_a^b F'(x)\,dx = F(b) - F(a).$$

i.e. "sum"

iv. Prove the Second Fundamental Theorem (Differentiating Integrals) which states that for a continuous function $f : [a, b] \to \mathbb{R}$,

$$\frac{d}{dx}\left[\int_a^x f(t)\,dt\right] = f(x) \qquad \text{for all } x \text{ in the open interval } (a, b).$$

But 1st, an important definition...

Let a and b be real numbers with $a < b$. If n is a natural number and

$$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b,$$

then $P = \{x_0, \ldots, x_n\}$ is called a *partition* of the interval $[a, b]$. For each index $i \geq 0$, we call x_i a *partition point* of P, and if $i \geq 1$, we call the interval $[x_{i-1}, x_i]$ a *partition interval* of P. The crudest partition of $[a, b]$ occurs when $n = 1$, so that $x_0 = a$ and $x_1 = b$, in which case there are just two partition points and one partition interval of the partition P.

Recall Riemann Sums...

left sum: t_k is the left endpoint of $[x_{k-1}, x_k]$

right sum: t_k is the right endpoint of $[x_{k-1}, x_k]$

midpoint sum: t_k is the midpoint of $[x_{k-1}, x_k]$

Let $f(x) = x^2$ and let $[-1, 2]$ be partitioned by $P = \{-1, 0, 1, \frac{3}{2}, 2\}$. Find the three Riemann sums listed above.

Solution For each of the Riemann sums, the t_k are shown in Figure 5.13. Using that information, we find the following values:

$$\text{left sum: } \sum_{k=1}^{4} f(t_k)\,\Delta x_k = \overbrace{f(-1)}^{1}\cdot 1 + \overbrace{f(0)}^{0}\cdot 1 + \overbrace{f(1)}^{1}\cdot\frac{1}{2} + \overbrace{f\left(\frac{3}{2}\right)}^{9/4}\cdot\frac{1}{2} = \frac{21}{8}$$

$$\text{right sum: } \sum_{k=1}^{4} f(t_k)\,\Delta x_k = \overbrace{f(0)}^{0}\cdot 1 + \overbrace{f(1)}^{1}\cdot 1 + \overbrace{f\left(\frac{3}{2}\right)}^{9/4}\cdot\frac{1}{2} + \overbrace{f(2)}^{4}\cdot\frac{1}{2} = \frac{33}{8}$$

$$\text{midpoint sum: } \sum_{k=1}^{4} f(t_k)\,\Delta x_k = \overbrace{f\left(-\frac{1}{2}\right)}^{1/4}\cdot 1 + \overbrace{f\left(\frac{1}{2}\right)}^{1/4}\cdot 1 + \overbrace{f\left(\frac{5}{4}\right)}^{25/16}\cdot\frac{1}{2} + \overbrace{f\left(\frac{7}{4}\right)}^{49/16}\cdot\frac{1}{2}$$

$$= \frac{45}{16} \quad \square$$

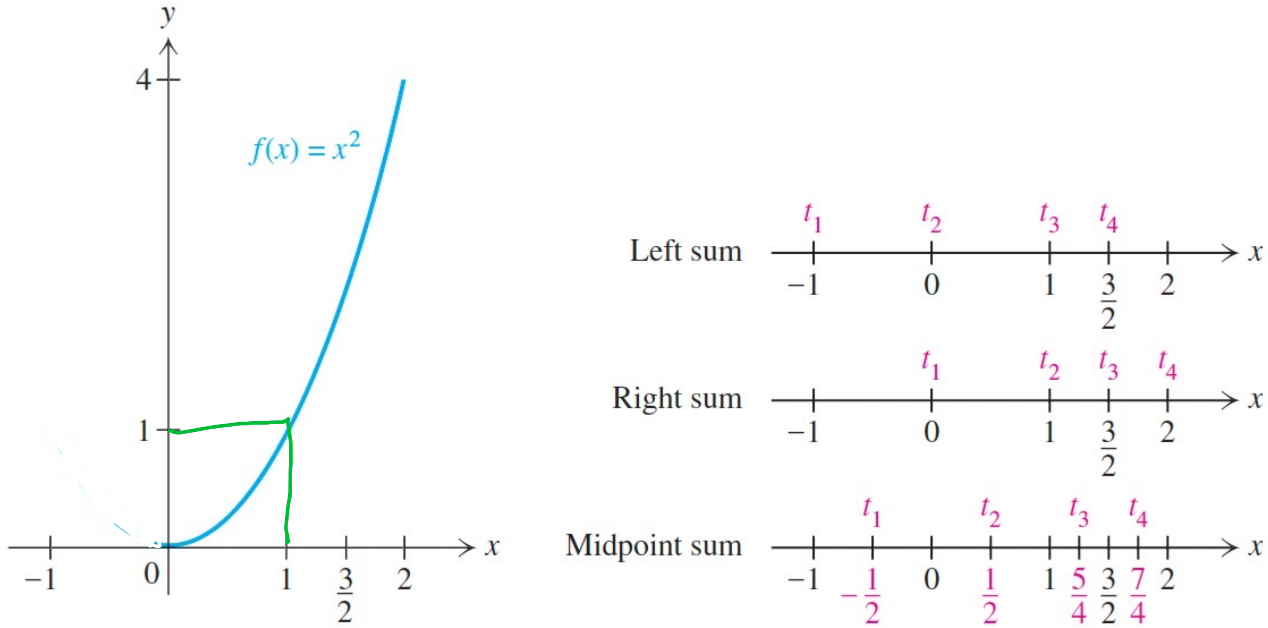

FIGURE 5.13

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Suppose that the function $f : [a, b] \to \mathbb{R}$ is bounded and that $P = \{x_0, \ldots, x_n\}$ is a partition of its domain $[a, b]$. For each index $i \geq 1$, we define

→ glb

$$\begin{cases} m_i \equiv \inf\{f(x) \mid x \text{ in } [x_{i-1}, x_i]\} \\ \qquad\qquad \text{and} \\ M_i \equiv \sup\{f(x) \mid x \text{ in } [x_{i-1}, x_i]\}. \end{cases} \tag{6.1}$$

→ lub

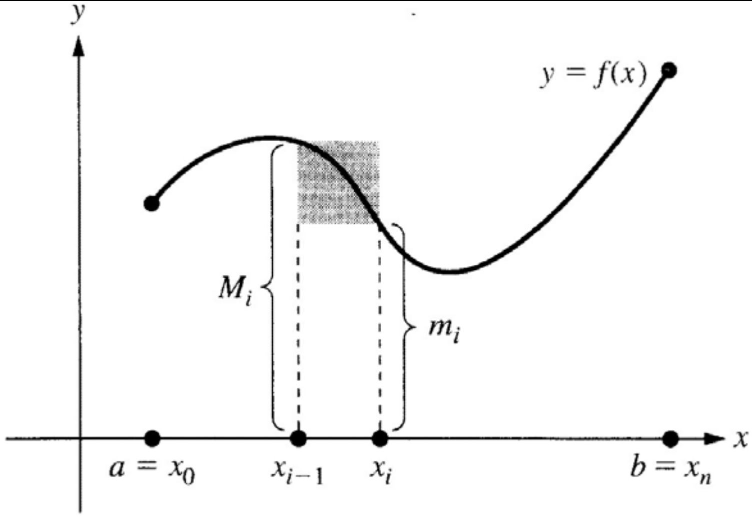

FIGURE 6.2 Approximation of the area over the partition interval $[x_{i-1}, x_i]$.

We then define

→ Lower Darboux Sum

$$\begin{cases} L(f, P) \equiv \displaystyle\sum_{i=1}^{n} m_i(x_i - x_{i-1}) \\ \qquad\qquad \text{and} \\ U(f, P) \equiv \displaystyle\sum_{i=1}^{n} M_i(x_i - x_{i-1}). \end{cases}$$

→ Upper Darboux Sum

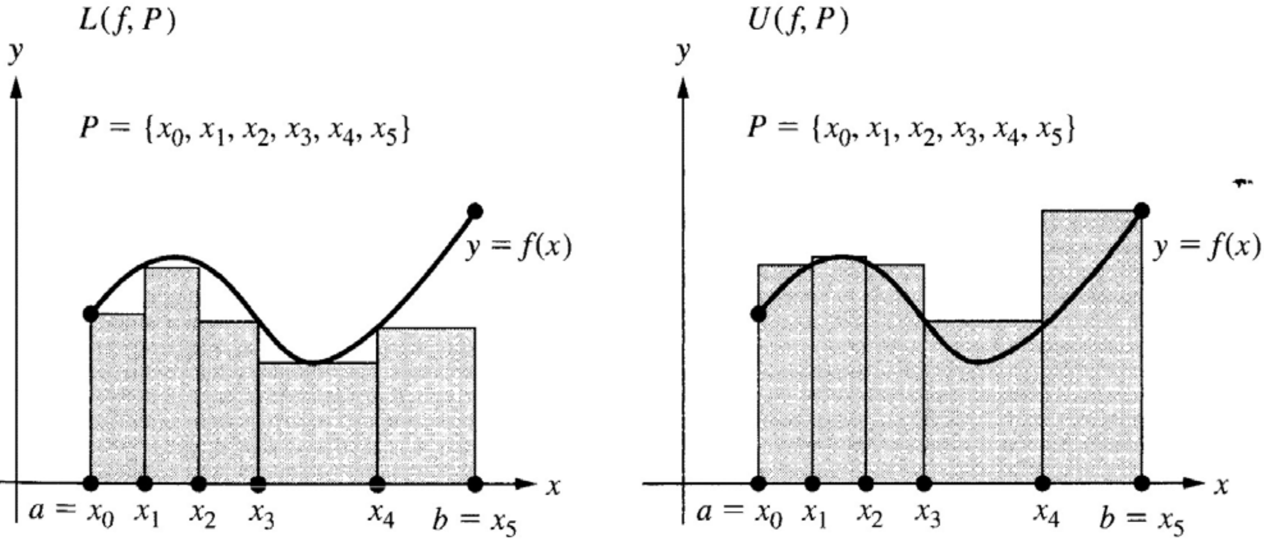

FIGURE 6.3 Upper and lower Darboux sums.

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* We will frequently use the fact that if $P = \{x_0, \ldots, x_n\}$ is a partition of the interval $[a, b]$, then the length of the interval $[a, b]$ is the sum of the lengths of the partition intervals of P; that is,

$$b - a = \sum_{i=1}^{n} (x_i - x_{i-1}). = (\cancel{x_1} - x_0) + (x_2 - \cancel{x_1}) + \cdots (x_n - x_{n-1}) \tag{6.5}$$

↳ Why?

Let's put the new notation into practice!

Lemma 6.1 *Suppose that the function $f : [a, b] \to \mathbb{R}$ is bounded and the numbers m and M have the property that*

$$\underline{\;m\;} \leq f(x) \leq \underline{\;M\;} \quad \text{for all } x \in [a, b].$$

Then, if P is a partition of the domain $[a, b]$,

= upper-Darboux sum of f with partition

$$m(b - a) \leq \underline{L(f,P)} \quad \text{and} \quad \underline{U(f,P)} \leq M(b - a).$$

Proof.
Let $P = \{\underline{x_0, x_1, \ldots x_n}\}$. For each index $i \geq 1$, the number m is a lower bound for the set $\{f(x) \mid x \in [x_{i-1}, x_i]\}$, so that, by the definition of infimum,

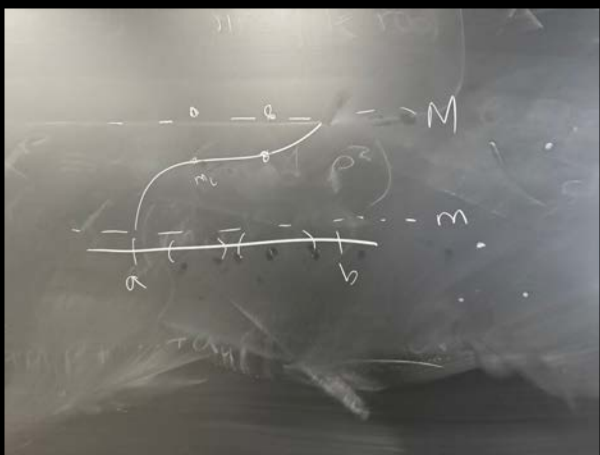
$$m \leq \underline{m_i}.$$

Thus, by the sum of lengths formula (6.5),

$$\begin{aligned} m(b - a) &= m \sum_{i=1}^{n} \underline{(x_i - x_{i-1})}, \\ &= \sum_{i=1}^{n} m \underline{(x_i - x_{i-1})}, \\ &\leq \sum_{i=1}^{n} \underline{m_i} (\underline{x_i - x_{i-1}}), \\ &= \underline{L(f,P)}. \end{aligned}$$

Therefore, $m(b - a) \leq L(f, P)$. By a similar argument, $\underline{U(f,P)} \leq M(b - a)$. ■

Intuition Guiding Diagram →



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Lemma 6.2 The Refinement Lemma Suppose that the function $f : [a, b] \to \mathbb{R}$ is bounded and that P is a partition of its domain $[a, b]$. If P^* is a refinement of P, then

ex) $P = \{0, \frac{1}{2}, 1\}$, $P^* = \{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\}$, i.e., $P \subseteq P^*$

$$L(f, P) \leq L(f, P^*) \quad \text{and} \quad U(f, P^*) \leq U(f, P).$$

* Given two partitions P_1 and P_2 of the interval $[a, b]$, the partition P^* formed by taking the union of the partition points of P_1 and of P_2 is a *common refinement* of P_1 and P_2 since P^* is a refinement of both P_1 and P_2. * $P^* = P_1 \cup P_2$

Moving towards our goal of defining an integral...

Upper and Lower Integrals

Definition Suppose that the function $f : [a, b] \to \mathbb{R}$ is bounded. Then we define the lower integral of f on $[a, b]$, which we denote by $\underline{\int}_a^b f$, by

$$\underline{\int_a^b} f \equiv \underbrace{\overset{\text{least upper bound}}{\sup}}_{\text{notice we don't say minimum as we may not attain a min!}}\{L(f, P) \mid P \text{ a partition of the interval } [a, b]\}. \tag{6.8}$$

We define the upper integral of f on $[a, b]$, which we denote by $\overline{\int}_a^b f$, by

$$\overline{\int_a^b} f \equiv \underbrace{\overset{\text{greatest lower bound}}{\inf}}_{\text{same logic for not using max}}\{U(f, P) \mid P \text{ a partition of the interval } [a, b]\}. \tag{6.9}$$

Example:

a sequence of partitions with $n \to \infty$

Suppose $L(f, P_n) = \{1, 2, 3\}$, and

$U(f, P_n) = \{4, 5, 6\}$.

generally Darboux sums are very difficult to calculate

Then $\underline{\int_a^b} f = 3$ and $\overline{\int_a^b} f = 4$

The idea is we get $L(f, P_n)$ and $U(f, P_n)$ as close as possible to the actual $\int_a^b f$ (i.e., minimize error).

Exam #2 Review

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1) Let $f(x) = x^3 - 2x^2 - x - 1$. Prove that there is a number c such that $f(c) = 1$.

2) Let $g(x) = \sin 1/x$ for $0 < x < 1$. Prove that g is not uniformly continuous on $(0, 1)$.

3) Let $f : R \to R$ such that $-x^2 \leq f(x) \leq x^2$ for all x. Prove by using the definition of the derivative that $f'(0)$ exists, and that $f'(0) = 0$.

4) ~~Let $h(x) = \sqrt{4+x}$. Show that h^{-1} exists, and find the domain of h^{-1}~~. Then find $(h^{-1})'(3)$.

5) For $c > 0$, prove that the following equation does not have two solutions:

$$x^3 - 3x + c = 0, \qquad 0 < x < 1.$$

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c) Let $f(x) = x^3 - 2x^2 - x - 1$. Prove that there is a number c such that $f(c) = 1$.

SCRAP

- IVT
- f is continuous
- $f(-1) = (-1)^3 - 2(-1)^2 - (-1) - 1$
  $= -1 - 2 + 1 - 1$
  $= -3$

- $f(10) = (10)^3 - 2(10)^2 - (10) - 1$
  $= 1000 - 200 - 10 - 1$
  $= 789$

Proof: Note that f is a polynomial, so f is continuous on $\mathbb{R}$. Then by the IVT, with $f(-1) = -3$ and $f(10) = 789$, there exists a value c in $(-1, 10)$ such that $f(c) = 1$, since $f(-1) < 1 < f(10)$. ∎

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2) Let $g(x) = \sin 1/x$ for $0 < x < 1$. Prove that g is not uniformly continuous on $(0, 1)$.

SCRAP

Let $a_n, b_n \in (0,1)$. If $\lim_{n\to\infty}[a_n - b_n] = 0$, then $\lim_{n\to\infty}[g(a_n) - g(b_n)] = 0$.

Uniformly continuous

Proof:

Let $a_n = \dfrac{1}{\frac{\pi}{2} + 2n\pi}$ and $b_n = \dfrac{1}{\frac{3\pi}{2} + 2n\pi}$, so

$a_n, b_n \in (0,1)$, and $\lim_{n\to\infty}[a_n - b_n] = 0$.

Then by the periodicity of sine, we have the following:

$$g(a_n) = \sin\left(\frac{\pi}{2} + 2n\pi\right) = \sin\left(\frac{\pi}{2}\right) = 1$$

and

$$g(b_n) = \sin\left(\frac{3\pi}{2} + 2n\pi\right) = \sin\left(\frac{3\pi}{2}\right) = -1.$$

Thus, $\lim_{n\to\infty} g(a_n) - g(b_n) = 1 - (-1) = 2 \neq 0$,

so $\sin\left(\frac{1}{x}\right)$ is <u>not</u> uniformly continuous. ∎

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$|f(x)| \leq x^2$

3) Let $f : R \to R$ such that $-x^2 \leq f(x) \leq x^2$ for all x. Prove by using the definition of the derivative that $f'(0)$ exists, and that $f'(0) = 0$.

If $-x^2 \leq f(x) \leq x^2$, then

$-x^2 - f(0) \leq f(x) - f(0) \leq x^2 - f(0)$.

Assume $x > 0$, without loss of generality.

Then:

$$\frac{-x^2 - f(0)}{x} \leq \frac{f(x) - f(0)}{x} \leq \frac{x^2 - f(0)}{x}$$

But $-(0)^2 \leq f(0) \leq (0)^2$, so $f(0) = 0$.

Thus, $\frac{-x^2 - f(0)}{x} = \frac{-x^2}{x} = -x$ (recall $x > 0$, so $x \neq 0$),

and $\frac{x^2 - f(0)}{x} = x$.

In turn, $-x \leq \frac{f(x) - f(0)}{x} \leq x$.

Observe that $\lim_{x \to 0} -x = \lim_{x \to 0} x = 0$

(by continuity of polynomials),

so by comparison, $\lim_{x \to 0} \frac{f(x) - f(0)}{x} = 0$.

But this is the definition of $f'(0)$.

So $f'(0) = 0$, and hence, exists.

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not a question you'll be asked on Exam #2

4) ~~Let $h(x) = \sqrt{4+x}$. Show that h^{-1} exists, and find the domain of h^{-1}.~~ Then find $(h^{-1})'(3)$.

$*$ $h(x) = (4+x)^{1/2}$, so $h'(x) = \dfrac{1}{2\sqrt{4+x}}$

$*$ Note that $h(5) = \sqrt{4+5} = \sqrt{9} = 3$,

so $h^{-1}(3) = 5$.

Thus, $(h^{-1})'(3) = \dfrac{1}{h'(5)} = \dfrac{1}{\left(\dfrac{1}{2\sqrt{4+5}}\right)} = \dfrac{1}{\left(\dfrac{1}{2\sqrt{9}}\right)} = 6$

5)

For $c > 0$, prove that the following equation does not have two solutions:

$$x^3 - 3x + c = 0, \qquad 0 < x < 1.$$

Suppose there are two solutions, that is, for $h(x) = x^3 - 3x + c$, we have that there exists x_1 and x_2 in $(0,1)$ such that $h(x_1) = h(x_2) = 0$. By Rolle's Theorem, noting h is continuous, we have that there exists $x_* \in (x_1, x_2)$ such that $f'(x_*) = 0$. Note that $f'(x_*) = 3x_*^2 - 3$, so we have $3x_*^2 - 3 = 0$, or $x_* = \pm 1$.

So either $x_* = 1$ or $x_* = -1$, a contradiction to $x_* \in (0,1)$. ∎

The Archimedes-Riemann Theorem (6.2)

Definition Suppose that the function $f:[a,\, b] \to \mathbb{R}$ is bounded. Then we say that $f:[a,\, b] \to \mathbb{R}$ is *integrable*, or that f is integrable on $[a,\, b]$, provided that

$$\underline{\int_a^b} f = \overline{\int_a^b} f.$$

When this is so, the integral of the function $f:[a,\, b] \to \mathbb{R}$, denoted by $\int_a^b f$, is defined by

$$\int_a^b f \equiv \underline{\int_a^b} f = \overline{\int_a^b} f.$$

(ART)

Theorem 6.8 The Archimedes–Riemann Theorem[3] Let $f:[a,\, b] \to \mathbb{R}$ be a bounded function. Then f is integrable on $[a,\, b]$ if and only if there is a sequence $\{P_n\}$ of partitions of the interval $[a,\, b]$ such that

$$\lim_{n\to\infty} [U(f,\, P_n) - L(f,\, P_n)] = 0. \tag{6.15}$$

Moreover, for any such sequence of partitions,

$$\lim_{n\to\infty} L(f,\, P_n) = \int_a^b f \quad \text{and} \quad \lim_{n\to\infty} U(f,\, P_n) = \int_a^b f. \tag{6.16}$$

Definition Let the function $[a,\, b]:\mathbb{R} \to \mathbb{R}$ be bounded and for each natural number n let P_n be a partition of its domain $[a,\, b]$. Then $\{P_n\}$ is said to be an *Archimedean sequence of partitions* for f on $[a,\, b]$ provided that

$$\lim_{n\to\infty} [U(f,\, P_n) - L(f,\, P_n)] = 0.$$

Proof Recall

used in proof of ART

Lemma 6.7. *For a bounded function $f : [a,b] \to \mathbb{R}$ and a partition P of $[a,b]$,*

$$L(f,P) \le \underline{\int_a^b} f \le \overline{\int_a^b} f \le U(f,P).$$

As a consequence, we also have the following three inequalities:

$$0 \le \overline{\int_a^b} f - \underline{\int_a^b} f \le U(f,P) - L(f,P),$$

$$0 \le U(f,P) - \overline{\int_a^b} f \le U(f,P) - L(f,P),$$

$$0 \le \underline{\int_a^b} f - L(f,P) \le U(f,P) - L(f,P).$$

Proof.
Since the lower integral is an upper bound for the collection of lower ~~Riemann~~ Darboux sums and the upper integral is a lower bound for the collection of upper Darboux sums, we have

$$L(f,P) \le \underline{\int_a^b} f \quad \text{and} \quad \overline{\int_a^b} f \le U(f,P).$$

But, according to Lemma 6.4[2],

$$\underline{\int_a^b} f \le \overline{\int_a^b} f.$$

Therefore, the first inequality holds, and the last three inequalities follow immediately from this one. ■

[2] **Lemma 6.4.** *For a bounded function $f : [a,b] \to \mathbb{R}$,*

$$\underline{\int_a^b} f \le \overline{\int_a^b} f.$$

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A useful type of partition...

Regular Partitions

Definition For a natural number n, the partition $P = \{x_0, \ldots, x_n\}$ of the interval $[a, b]$ defined by

$$x_i = a + i\frac{(b-a)}{n} \qquad \text{for } 0 \le i \le n$$

is called the *regular partition* of $[a, b]$ into n partition intervals.

A regular partition of $[a, b]$ into n partition intervals is characterized by the fact that all of its partition intervals have the same length, namely, $(b-a)/n$.

An interesting characterization comparing each interval of a partition...

The Gap of a Partition

Definition For a partition $P = \{x_0, \ldots, x_n\}$ of the interval $[a, b]$, we define the *gap of* P, denoted by gap P, to be the length of the largest partition interval of P; that is,

$$\text{gap } P \equiv \max_{1 \le i \le n} [x_i - x_{i-1}].$$

Observe that for a partition P and a positive number ϵ, gap $P < \epsilon$ if and only if each partition interval of P has length less than ϵ.

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*Other important results/facts:

1) A monotonically increasing function $f:[a,b]\to\mathbb{R}$ is integrable.

*notice continuity isn't necessary!, Why?

2) A step function $[a,b]\to\mathbb{R}$ is integrable.

Definition A function $f:[a,\ b]\to\mathbb{R}$ is called a *step function* provided that there is a partition $P^* = \{z_0, \ldots, z_k\}$ of its domain $[a,\ b]$ and numbers $c_1, \ldots, c_k$ such that for $1 \le i \le k$,

$$f(x) = c_i \qquad \text{for all } x \text{ in the open partition interval } (z_{i-1},\ z_i).$$

3)

Leibnitz Notation

For an integrable function $f:[a,\ b]\to\mathbb{R}$, we have denoted the value of the integral by the symbol $\int_a^b f$. The value of the integral is also often denoted by symbols such as

$$\int_a^b f(x)\,dx \qquad \text{or} \qquad \int_a^b f(t)\,dt.$$

4) Sum of 1st n integers: $\displaystyle\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

5) Sum of 1st n squares: $$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

useful for HW!

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Additivity, Monotonicity, and Linearity (6.3)

Theorem 6.12 Additivity over Intervals Let the function $f : [a, b] \to \mathbb{R}$ be integrable on $[a, b]$ and let c be a point in the open interval (a, b). The f is integrable both on $[a, c]$ and on $[c, b]$, and furthermore,

$$\int_a^b f = \int_a^c f + \int_c^b f. \tag{6.27}$$

Theorem 6.13 Monotonicity of the Integral Suppose that the functions $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ are integrable and that

$$f(x) \le g(x) \qquad \text{for all } x \text{ in } [a, b].$$

Then

$$\int_a^b f \le \int_a^b g.$$

Proof Recall:

Theorem 6.13 (Monotonicity of the Integral). *Suppose that the functions $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ are integrable and that*

$$f(x) \le g(x) \quad \text{for all } x \in [a, b].$$

Then

$$\int_a^b f \le \int_a^b g.$$

Proof.

→ Archimedes-Riemann Theorem

By the ART and the Refinement lemma, there is a sequence $\{P_n\}$ of partitions of the interval $[a, b]$ that is both an Archimedean sequence of partitions for f on $[a, b]$ and for g on $[a, b]$. Therefore,

$$\lim_{n\to\infty} L(f, P_n) = \int_a^b f \quad \text{and} \quad \lim_{n\to\infty} U(g, P_n) = \int_a^b g.$$

also could say $\lim_{n\to\infty} U(f, P_n) = \int^b f$, but this isn't helpful in our proof

Since

$$f(x) \le g(x) \quad \text{for all } x \in [a, b],$$

it follows directly from the definition of the Upper Darboux Sum that for each index n,

$$L(f, P_n) \le U(g, P_n).$$

→ because P_n is a common refinement, and m_i for f is less than or equal to M_i for g on each interval (since $f \le g$ everywhere)

note typo in book

Thus,

$$\int_a^b f = \lim_{n\to\infty} L(f, P_n) \le \lim_{n\to\infty} U(g, P_n) = \int_a^b g.$$

∎

*Useful in proof of linearity of integral

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Lemma 6.14 Let $f:[a,b]\to\mathbb{R}$ and $g:[a,b]\to\mathbb{R}$ be bounded functions and let P be a partition of their domain $[a,b]$. Then

$$L(f,P)+L(g,P)\le L(f+g,P) \quad \text{and} \quad U(f+g,P)\le L(f,P)+U(g,P). \tag{6.30}$$

Moreover, for any number α,

$$\begin{aligned} U(\alpha f,P)&=\alpha U(f,P) &\text{and}\quad L(\alpha f,P)&=\alpha L(f,P) &&\text{if } \alpha\ge 0\\ U(\alpha f,P)&=\alpha L(f,P) &\text{and}\quad L(\alpha f,P)&=\alpha U(f,P) &&\text{if } \alpha<0. \end{aligned} \tag{6.31}$$

*What's the intuition here?

Theorem 6.15 Linearity of the Integral Let the functions $f:[a,b]\to\mathbb{R}$ and $g:[a,b]\to\mathbb{R}$ be integrable. Then for any two numbers α and β, the function $\alpha f+\beta g:[a,b]\to\mathbb{R}$ is integrable and

$$\int_a^b[\alpha f+\beta g]=\alpha\int_a^b f+\beta\int_a^b g. \tag{6.33}$$

Corollary 6.16 Suppose that the functions $f:[a,b]\to\mathbb{R}$ and $|f|:[a,b]\to\mathbb{R}$ are integrable. Then

$$\left|\int_a^b f(x)\,dx\right|\le\int_a^b|f(x)|\,dx. \tag{6.35}$$

*Aligns nicely with our geometric interpretation of an integral as the area under a curve!

(Continuity and Integrability (6.4))

Our primary goal in this section is to show that a continuous function on a closed bounded interval is integrable. To do so, it is useful to first isolate the following estimate of the difference of upper and lower Darboux sums for a continuous function.

Lemma 6.17 Let the function $f : [a, b] \to \mathbb{R}$ be continuous and let P be a partition of its domain $[a, b]$. Then there is a partition interval of P that contains two points u and v for which the following estimate holds:

→ reminds us of unif. continuity!

$$0 \le U(f, P) - L(f, P) \le [f(v) - f(u)][b - a]. \tag{6.36}$$

Theorem 6.18 A continuous function on a closed bounded interval, $f : [a, b] \to \mathbb{R}$, is integrable.

*End of Proof →

This limit, together with the inequality (6.38), implies that

$$0 \le \lim_{n\to\infty} [U(f, P_n) - L(f, P_n)] \le \lim_{n\to\infty} [f(v_n) - f(u_n)][b - a] = 0.$$

Thus, the sequence $\{P_n\}$ is an Archimedean sequence of partitions for f on $[a, b]$. According to the Archimedes–Riemann Theorem, f is integrable on $[a, b]$. ■

*Useful fact in proof →

Recall that Theorem 3.17 asserts that a continuous function on a closed bounded interval, $f : [a, b] \to \mathbb{R}$, is uniformly continuous; that is, for any two sequences $\{u_n\}$ and $\{v_n\}$ in its domain $[a, b]$,

$$\lim_{n\to\infty} [f(u_n) - f(v_n)] = 0 \qquad \text{if } \lim_{n\to\infty} [u_n - v_n] = 0.$$

This is the property of a continuous function on a closed bounded interval that implies that it is integrable.

Theorem 6.19 Suppose that the function $f : [a, b] \to \mathbb{R}$ is bounded on the closed interval $[a, b]$ and is continuous on the open interval (a, b). Then f is integrable on $[a, b]$ and the value of the integral, $\int_a^b f$, does not depend on the values of f at the endpoints of the interval.

Agrees with intuition and our geometric interpretation of an integral giving the area under a curve!

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1st Fundamental Theorem of Calculus: Integrating Derivatives (6.5)

WT:

Calculate $\frac{d}{dx}\int_{t=0}^{t=x^2} t^2\,dt$.

"dummy variable"

function of x

Step 1: $\int_0^{x^2} t^2\,dt = \left[\frac{t^3}{3}\right]_0^{x^2} = \left[\frac{x^2 \cdot x^3}{3}\right] - \left[\frac{x^2(0)}{3}\right] = \frac{x^5}{3}$

0

Step 2: $\frac{d}{dx}\frac{x^5}{3} = \boxed{\frac{5x^4}{3}}$

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1st Fundamental Theorem of Calculus: Integrating Derivatives (6.5)

(FTC)

Theorem 6.22 The First Fundamental Theorem: Integrating Derivatives Let the function $F : [a, b] \to \mathbb{R}$ be continuous on the closed interval $[a, b]$ and be differentiable on the open interval (a, b). Moreover, suppose that its derivative

$$F' : (a, b) \to \mathbb{R} \text{ is both continuous and bounded.}$$

Then

"Definite Integral"

$$* \quad \int_a^b F'(x)\,dx = F(b) - F(a).$$

Proof:

The function $F' : (a,b) \to \mathbb{R}$ is continuous and bounded. Thus, by Theorem 6.19, it is integrable on $[a,b]$ in the sense that any extension of F' to the closed interval $[a,b]$ is integrable, and the value of the resulting integral does not depend on the values of the extension at the endpoints of the interval.

see * on next page

By the preceding lemma, in order to verify the above formula (*), it suffices to verify that for each partition P on $[a,b]$, we have the following:

$$L(F',P) \le F(b) - F(a) \le U(F',P)$$

*note we aren't assuming this, but want to show it

Let $P = \{x_0, x_1, \dots, x_n\}$ be a partition of $[a,b]$. Fix an index $i \ge 1$. By assumption, $F : [x_{i-1}, x_i] \to \mathbb{R}$ is continuous on $[x_{i-1}, x_i]$. Also, F is differentiable on (x_{i-1}, x_i). By the MVT, there is a point c_i in (x_{i-1}, x_i) such that $F'(c_i) = \dfrac{F(x_i) - F(x_{i-1})}{x_i - x_{i-1}}$, or

$$F'(c_i)(x_i - x_{i-1}) = F(x_i) - F(x_{i-1}).$$

Note that $m_i = \inf\{F'(x) \mid x \in [x_{i-1}, x_i]\} \le F'(c_i) \le \sup\{F'(x) \mid x \in [x_{i-1}, x_i]\} = M_i$

continued...

Multiplying this last inequality by $x_i - x_{i-1}$ and substituting the Mean Value Formula (6.46), we obtain

$\rightarrow F'(c_i)(x_i - x_{i-1})$

$$m_i(x_i - x_{i-1}) \le F(x_i) - F(x_{i-1}) \le M_i(x_i - x_{i-1}).$$

Summing these n inequalities, we obtain the following inequality:

$$\sum_{i=1}^{n} m_i(x_i - x_{i-1}) \le \sum_{i=1}^{n} [F(x_i) - F(x_{i-1})] \le \sum_{i=1}^{n} M_i(x_i - x_{i-1}).$$

The left-hand sum is $L(f, P)$, the right-hand sum is $U(f, P)$, and moreover,

$$\sum_{i=1}^{n} [F(x_i) - F(x_{i-1})] = F(b) - F(a).$$

Lemma 6.21 Suppose that the function $f : [a, b] \to \mathbb{R}$ is integrable and that the number A has the property that for every partition P of $[a, b]$,

$$L(f, P) \le A \le U(f, P).$$

Then

$$\int_a^b f = A.$$

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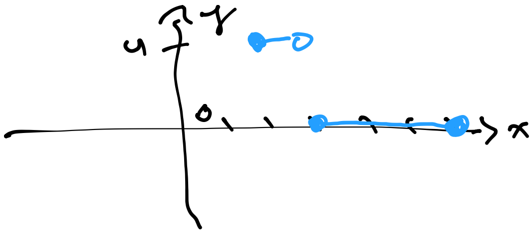
An interesting example:

Example 6.25 Define



$$f(x) = \begin{cases} 4 & \text{if } 2 \le x < 3 \\ 0 & \text{if } 3 \le x \le 6. \end{cases}$$

There is now no possibility of applying the First Fundamental Theorem (Integrating Derivatives) in the evaluation of $\int_2^6 f$ since the function $f : [2, 6] \to \mathbb{R}$ does not have an antiderivative (Exercise 4). Since this function is a step function, it is integrable. It is not difficult to see that $\int_2^6 f = 4$ (Exercise 3). ∎

4. Define

$$f(x) = \begin{cases} 4 & \text{if } 2 \le x < 3 \\ 0 & \text{if } 3 \le x \le 6. \end{cases}$$

a. Use the Identity Criterion to show that if there is an antiderivative $F : [2, 6] \to \mathbb{R}$ for f, then there are numbers c_1 and c_2 such that

$$F(x) = \begin{cases} 4x + c_1 & \text{if } 2 \le x < 3 \\ c_2 & \text{if } 3 \le x \le 6. \end{cases}$$

(a) If indeed f has an antiderivative F, then $F'(x) = 4$ for $2 \le x < 3$. Since the function $H(x) = 4x$ also has this property, it follows from the Identity Criterion that $F(x) = 4x + c_1$ for some real number c_1.

Similarly, note that if $H(x) = 0$, then $H'(x) = 0$, so by the Identity Criterion, $F(x) = c_2$ for some real number c_2.

$H' = F'$

$H = F + c$

* **Proposition 4.20 The Identity Criterion** Let I be an open interval and let the functions $g : I \to \mathbb{R}$ and $h : I \to \mathbb{R}$ be differentiable. Then these functions differ by a constant if and only if

$$g'(x) = h'(x) \qquad \text{for all } x \text{ in } I. \tag{4.10}$$

In particular, these functions are identically equal if and only if (4.10) holds and there is some point x_0 in I at which

$$g(x_0) = h(x_0).$$

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4. Define

$$f(x) = \begin{cases} 4 & \text{if } 2 \le x < 3 \\ 0 & \text{if } 3 \le x \le 6. \end{cases}$$

a. Use the Identity Criterion to show that if there is an antiderivative $F : [2, 6] \to \mathbb{R}$ for f, then there are numbers c_1 and c_2 such that

$$F(x) = \begin{cases} 4x + c_1 & \text{if } 2 \le x < 3 \\ c_2 & \text{if } 3 \le x \le 6. \end{cases}$$

b. From (a) show that by the continuity of F at $x = 3$, $12 + c_1 = c_2$.

By the continuity of F on $[2,6]$ (i.e., F' exists since it's an antiderivative, and since differentiability implies continuity, F is continuous), we know

$\lim_{x\to 3, x<3} F(x) = \lim_{x \to 3, x<3} 4x+c_1 = 12+c_1$, for $x \in [2,3)$.

Similarly, $\lim_{x\to 3, x>3} c_2 = c_2$, for $x \in [3,6]$.

That is, by continuity, it must be that $12+c_1=c_2$.

c. Show that the function F cannot be differentiable at $x = 3$ and therefore that the function f does not have an antiderivative.

For $x \in [2,6]$, with $x \neq 3$, by part b, we have that $\frac{F(x)-F(3)}{x-3} = \frac{\overset{F(x)}{4x+c_1} - c_2}{x-3} = \frac{4x-12}{x-3}$ if $x<3$, and $\frac{c_2-c_2}{x-3} = 0$ if $x>3$. Thus, $F'(3)$ does not exist since $\lim_{x\to 3, x<3} \frac{F(x)-F(3)}{x-3} = \lim_{x\to 3, x<3} \frac{4x-12}{x-3} \neq \lim_{x \to 3, x>3} 0 = \lim_{x\to 3, x>3} \frac{F(x)-F(3)}{x-3}$.

But F is supposed to be differentiable (a contradiction). Hence, F does not have an antiderivative. ∎

*By an antiderivative of f we mean a continuous function $F : [a, b] \to \mathbb{R}$ having a derivative on the open interval (a, b) such that

$$F'(x) = f(x) \qquad \text{for all } x \text{ in } (a, b). \tag{6.48}$$

* The above examples illustrate both the power and the limitations of the First Fundamental Theorem (Integrating Derivatives). It replaces the problem of calculating $\int_a^b f$ with the problem of finding an antiderivative for f. Frequently one can recognize an antiderivative, but there are cases when an antiderivative is not readily recognizable and there are cases where there is no antiderivative.

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2^{nd} Fundamental Theorem of Calculus: Differentiating Integrals (6.6)

*An interesting "version" of the MVT!

Theorem 6.26 The Mean Value Theorem for Integrals Suppose that the function $f : [a, b] \to \mathbb{R}$ is continuous. Then there is a point x_0 in the interval $[a, b]$ at which

$$\frac{1}{b-a}\int_a^b f = f(x_0).$$

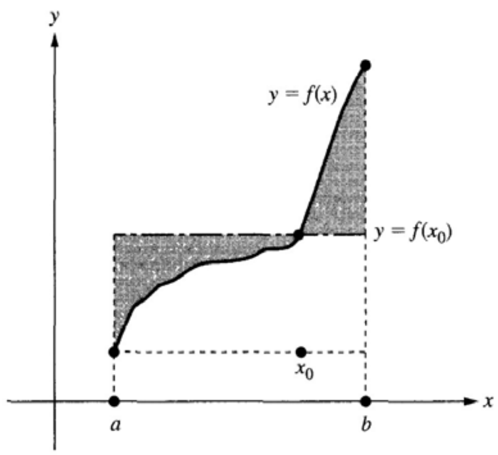

FIGURE 6.6 $f(x_0) = \dfrac{1}{b-a}\displaystyle\int_a^b f.$

*Interpreting an integral:

Proposition 6.27 Suppose that the function $f : [a, b] \to \mathbb{R}$ is integrable. Define

replace w/ continuous to get FTC!

$$F(x) = \int_a^x f \qquad \text{for all } x \text{ in } [a, b].$$

*Remember: An integral is, itself, a function!

Then the function $F : [a, b] \to \mathbb{R}$ is continuous.

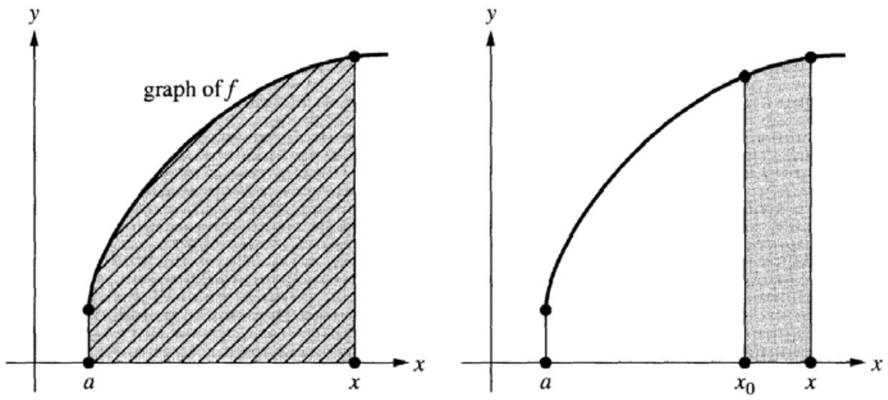

FIGURE 6.7 Shaded area equals $F(x) - F(x_0)$.

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Theorem 6.29 The Second Fundamental Theorem: Differentiating Integrals Suppose that the function $f:[a,b]\to\mathbb{R}$ is continuous. Then

$$\frac{d}{dx}\left[\int_a^x f\right] = f(x) \qquad \text{for all } x \text{ in } (a,b).$$

Let's analyze the Proof!

Define

$$F(x) \equiv \int_a^x f \qquad \text{for all } x \text{ in } [a,b].$$

↪ see previous proposition!

We have already verified that the function $F:[a,b]\to\mathbb{R}$ is properly defined and is, in fact, continuous. Let x_0 be a point in (a,b). We must show that

$$\lim_{x\to x_0}\frac{F(x)-F(x_0)}{x-x_0} = f(x_0).$$

$\to F'(x_0)$

Let x be a point in (a,b) with $x\neq x_0$. By the additivity over intervals property of the integral, Theorem 6.12,

$\int_a^x f$, $\int_a^{x_0} f$, so $\int_a^x f - \int_a^{x_0} f = \cancel{\int_a^{x_0} f} + \int_{x_0}^x f - \cancel{\int_a^{x_0} f} = \int_{x_0}^x f$

$$F(x)-F(x_0) = \int_{x_0}^x f \qquad \text{if } x > x_0,$$

while

varying x values between x_0 and x

$$F(x)-F(x_0) = -\int_x^{x_0} f \qquad \text{if } x < x_0.$$

*Similar idea

substitute

Consequently, applying the Mean Value Theorem for Integrals, we see that we can select a point $c(x)$ between x_0 and x such that

* $\frac{1}{x-x_0}\int_x^{x_0} f = f(c(x))$

$\frac{1}{x-x_0}[F(x)-F(x_0)] = f(c(x))$

$$\boxed{\frac{F(x)-F(x_0)}{x-x_0} = f(c(x)).} \tag{6.54}$$

→ See formula on previous page!

But the function $f:[a,b]\to\mathbb{R}$ is continuous at x_0, so that

$$\lim_{x\to x_0} f(c(x)) = f(x_0).$$

* As $x\to x_0$, $c(x)\to c(x_0)=x_0$, so by composition, $\lim_{x\to x_0} f(c(x)) = f(x_0)$

↳ a continuous function as captures all x values between x and x_0 by design

Thus,

$$F'(x_0) = \lim_{x\to x_0}\frac{F(x)-F(x_0)}{x-x_0} = \lim_{x\to x_0} f(c(x)) = f(x_0).$$

■

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Other Results/Facts:

1) **Definition** Let the function $f : [a, b] \to \mathbb{R}$ be integrable. Let c and d be numbers in $[a, b]$ such that $c < d$. We define

$$\int_d^c f \equiv -\int_c^d f \qquad \text{and} \qquad \int_c^c f = 0.$$

2) Let $f : I \to \mathbb{R}$ be integrable, for a closed-bounded interval I. Then for three points x_1, x_2, and x_3 in I:

$$\int_{x_1}^{x_3} f = \int_{x_1}^{x_2} f + \int_{x_2}^{x_3} f$$

3) **Corollary 6.31** Let I be an open interval and suppose that the function $f : I \to \mathbb{R}$ is continuous. Fix a point x_0 in I. Then

$$\frac{d}{dx}\left[\int_{x_0}^{x} f\right] = f(x) \qquad \text{for all } x \text{ in } I.$$

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Taylor Polynomials (8.1)

How could different degree polynomials better approximate $f(x)$?

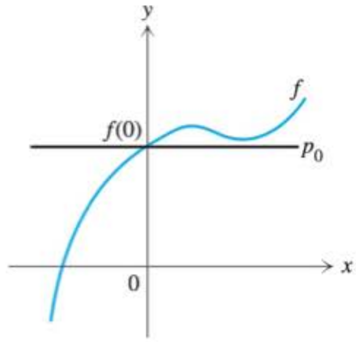


FIGURE 9.1

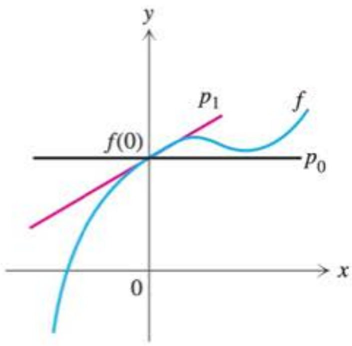
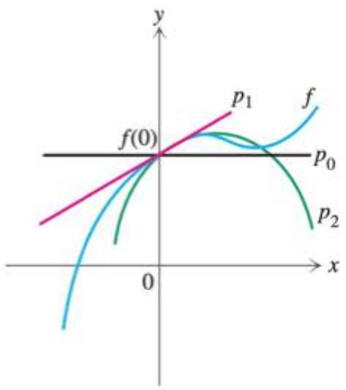


FIGURE 9.2



1st Taylor Polynomial

$$p_1(x) = f(0) + f'(0)x$$

$p_1(0) = f(0) + f'(0)(0) = f(0)$

$p_1'(0) = f'(0)$

$$p_2(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2$$

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Order of Contact of Two Functions

Definition Let I be a neighborhood of the point x_0. Two functions $f : I \to \mathbb{R}$ and $g : I \to \mathbb{R}$ are said to *have contact of order* 0 at x_0 provided that $f(x_0) = g(x_0)$. For a natural number n, the functions f and g are said to *have contact of order n* at x_0 provided that $f : I \to \mathbb{R}$ and $g : I \to \mathbb{R}$ have n derivatives and

$$f^{(k)}(x_0) = g^{(k)}(x_0) \qquad \text{for } 0 \le k \le n.$$

Proof Recall:

Proposition 8.2 *Let I be a neighborhood of the point x_0 and let n be a nonnegative integer. Suppose that the function $f : I \to \mathbb{R}$ has n derivatives. Then there is a unique polynomial of degree at most n that has contact of order n with the function $f : I \to \mathbb{R}$ at x_0. This polynomial is defined by the formula*

"Taylor Polynomial"

$$p_n(x) = f(x_0) + f'(x_0)(x - x_0) + \cdots + \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n.$$

Proof. If $n = 0$, then note that there is only one constant function whose value at x_0 is $\underline{f(x_0)}$.

So suppose that $n \ge 1$. From the differentiation of powers formula, it follows that

$$\frac{d^k}{dx^k}[p_n(x)]\Big|_{x=x_0} = \underline{f^{(k)}(x_0)} \qquad \text{for } 0 \le k \le n,$$

evaluate the k^{th} derivative of $p_n(x)$ at x_0

so the function f and the polynomial p_n have contact of order $\underline{n}$ at x_0.

It remains to prove uniqueness. However, if we take a general polynomial of degree at most n, written in powers of $x - x_0$ as

$$p(x) = c_0 + c_1(x - x_0) + \cdots + c_n(x - x_0)^n,$$

then, again from the differentiation of powers formula, we have that

$$\frac{d^k}{dx^k}[p(x)]\Big|_{x=x_0} = \underline{k! \cdot c_k} \qquad \text{for } 0 \le k \le n.$$

So that if the polynomial p has contact of order n with f, we must have $k!c_k = \underline{f^{(k)}(x_0)}$ for $0 \le k \le n$; that is, $p = \underline{p_n(x)}$. ■

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So what?

For two functions that have a high order of contact at a point, it is reasonable to expect that near this point the difference between the functional values will be small. In particular, if I is a neighborhood of the point x_0 and p_n is the nth Taylor polynomial for the function $f : I \to \mathbb{R}$ at x_0, one expects that for another point x in I, the difference $f(x) - p_n(x)$ can be estimated and shown to be small if x is close to x_0 and n is large.

What is really surprising is that frequently it happens that

$$\lim_{n\to\infty} [f(x) - p_n(x)] = 0,$$

even when the point x is far away from x_0. As we will show in Section 8.6, it can also happen that the Taylor polynomials for certain functions do not provide good approximations at any point x other than x_0, no matter how large the index[1] n. We define $R_n(x) \equiv f(x) - p_n(x)$ for all x in I, so that

$$f(x) = p_n(x) + R_n(x) \qquad \text{for all } x \text{ in } I,$$

and call $R_n : I \to \mathbb{R}$ the nth *remainder.* In Section 8.2, we will begin a rigorous analysis of this remainder.

Function and Series	First Few Terms	Interval of Convergence
* $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$	$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$	$(-\infty, \infty)$
* $\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$	$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$	$(-\infty, \infty)$
* $\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$	$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$	$(-\infty, \infty)$
$\ln x = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x-1)^n}{n}$	$(x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \cdots$	$(0, 2]$
* $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$	$1 + x + x^2 + x^3 + \cdots$	$(-1, 1)$
$(1+x)^k = \sum_{n=0}^{\infty} \frac{k(k-1)\cdots(k-(n-1))}{n!} x^n$	$1 + kx + \frac{k(k-1)}{2!} x^2 + \cdots$	$(-1, 1)^a$
$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$	$x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$	$[-1, 1]$

[a] Convergence at $x = \pm 1$ depends on the value of k.

Verification example

Find the Taylor Series for $f(x) = e^x$ about $x = 0$.

$$* \quad f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

$f^{(n)}(0)$

$f^{(0)}(x) = e^x \longrightarrow e^0 = 1$

$f^{(1)} \leftarrow f'(x) = e^x \longrightarrow e^0 = 1$

$f^{(2)} \leftarrow f''(x) = e^x \longrightarrow e^0 = 1$

$\vdots \qquad \vdots$

$f^{(n)}(x) = e^x \longrightarrow e^0 = 1$

For $f(x) = e^x$:

$$e^x = \frac{1 \cdot x^{0 \leftarrow n}}{0!} + \frac{1 \cdot x^1}{1!} + \frac{1 \cdot x^2}{2!} + \cdots$$

$0! \to n$

$f^{(0)}(0) \quad f^{(1)}(0) \quad f^{(2)}(0)$

$$= \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

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ex) Expand $g(x) = 4x^3 - 3x^2 + 5x - 1$ in terms of powers of $x-2$.

$\hookrightarrow (x-2)^n$

$\hookrightarrow$ i.e., centered at $x = 2$

$$f^{(n)}(2)$$

$$g^{(0)}(x) = 4x^3 - 3x^2 + 5x - 1 \rightarrow 29$$

$$g^{(1)}(x) = 12x^2 - 6x + 5 \rightarrow 41$$

$$g^{(2)}(x) = 24x - 6 \rightarrow 42$$

$$g^{(3)}(x) = 24 \rightarrow 24$$

$$g^{(4)}(x) = 0 \rightarrow 0$$

$$\vdots \qquad \vdots$$

$$\Rightarrow \text{So: } g(x) = \frac{29(x-2)^0}{\underbrace{0!}_{=1}} + \frac{41(x-2)^1}{\underbrace{1!}_{=1}} + \frac{42(x-2)^2}{\underbrace{2!}_{=2}} + \frac{24(x-2)^3}{\underbrace{3!}_{=6}} + 0 + 0 \cdots$$

$$= \boxed{29 + 41(x-2) + \overset{42/2}{21}(x-2)^2 + \overset{24/6}{4}(x-2)^3}$$

The Lagrange Remainder Theorem (8.2)

* Recall:

Lemma 8.7 Let I be an open interval and let n be a nonnegative integer and suppose that the function $f : I \to \mathbb{R}$ has $n+1$ derivatives. Suppose also that at the point x_0 in I,

$$f^{(k)}(x_0) = 0 \qquad \text{for } 0 \le k \le n.$$

Then for each point $x \neq x_0$ in I, there is a point c strictly between x and x_0 at which

$$f(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}.$$

The general remainder theorem is a simple extension of the above lemma.

Theorem 8.8 (The Lagrange Remainder Theorem) *Let I be a neighborhood of the point x_0 and let n be a nonnegative integer. Suppose that the function $f : I \to \mathbb{R}$ has $n+1$ derivatives. Then for each point $x \neq x_0$ in I, there is a point c strictly between x and x_0 such that*

$$f(x) = \underbrace{\sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k}_{P_n(x) \to \text{Taylor Polynomial}} + \underbrace{\frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}}_{R_n(x) \to \text{remainder}}$$

Proof
Consider the nth Taylor polynomial for the function f at x_0,

$$p_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k.$$

Since the functions f and p_n have contact of order n at x_0, it follows that if we define the function $R : I \to \mathbb{R}$ by

$$R(x) \equiv f(x) - p_n(x) \quad \text{for all } x \in I,$$

$R(x_0) = f(x_0) - P_n(x_0)$, $R'(x_0) = f'(x_0) - P_n'(x_0)$

then

$$R(x_0) = R'(x_0) = \cdots = R^{(n)}(x_0) = 0.$$

According to the preceding lemma, if $x \neq x_0$ is in I, then there is a point c strictly between x and x_0 such that

$$R(x) = \frac{R^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1} \to (x - x_0)^{n+1}$$

But since p_n is a polynomial of degree at most n, its $n+1$ derivative is identically 0, and therefore

$$R^{(n+1)}(c) = f^{(n+1)}(c) - p_n^{(n+1)}(c) = f^{(n+1)}(c).$$

From the previous equation it follows that

$$f(x) - p_n(x) = R(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}, \to \text{Lagrange Remainder}$$

and therefore we have established the formula (8.3). ■

* **Corollary 8.9** Suppose that p is a polynomial of degree at most n and let x_0 be any point. Then the nth Taylor polynomial for p at x_0 is p itself.

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The Convergence of Taylor Polynomials (8.3)

$\{s_0, s_1, s_2, \ldots, s_n\} = \{a_0, a_1 + a_2, \ldots\}$

For a sequence of numbers $\{a_k\}$ that is indexed by the nonnegative integers, we define

"nth partial sum" 1st partial sum

$$s_n = \sum_{k=0}^{n} a_k \qquad \text{for every nonnegative integer } n,$$

$s_0 = \sum_{k=0}^{0} a_k = a_0, \; s_1 = \sum_{k=0} a_k = a_0 + a_1$

and obtain a new sequence $\{s_n\}$. The sequence $\{s_n\}$ is called the *sequence of partial sums* for the series $\sum_{k=0}^{\infty} a_k$, and a_k is called the kth *term* of the series $\sum_{k=0}^{\infty} a_k$. We write

$$\sum_{k=0}^{\infty} a_k = \lim_{n \to \infty} \left[\sum_{k=0}^{n} a_k \right]$$

if the sequence $\{s_n\}$ converges. If the sequence $\{s_n\}$ does not converge, then we say that the series $\sum_{k=0}^{\infty} a_k$ *diverges*.

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k. \tag{8.11}$$

This formula is called a *Taylor series expansion* of the function $f : I \to \mathbb{R}$ about the point x_0. By its very definition, (8.11) holds at x if and only if

$$\lim_{n \to \infty} [f(x) - p_n(x)] = 0. \tag{8.12}$$

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Geometric Series

ex) $\frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \dots$

Formulas:

1) $$\sum_{k=0}^{n} r^k = \frac{1-r^{n+1}}{1-r}$$

$r \neq 1$

2) $$\sum_{k=0}^{\infty} r^k = \frac{1}{1-r}$$

*3) $$\sum_{k=m}^{n} r^k = \sum_{k=0}^{n} r^k - \sum_{k=0}^{m-1} r^k$$

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* How do we make sense of the following?

Lemma 8.13 For any number c,

useful fact for future limits

$$\lim_{n\to\infty} \frac{c^n}{n!} = 0.$$

i.e., $\frac{|c|}{k} < \frac{1}{2}$

Proof

Choose k to be a natural number such that $k \geq 2|c|$. Then if $n \geq k$,

$$\begin{aligned} 0 &\leq \left|\frac{c^n}{n!}\right| \\ &= \left[\frac{|c|}{1}\cdots\frac{|c|}{k}\right]\left[\frac{|c|}{k+1}\cdots\frac{|c|}{n}\right] \\ &\leq |c|^k\left(\frac{1}{2}\right)^{n-k} \\ &= |c|^k 2^k \left(\frac{1}{2}\right)^n. \end{aligned}$$

? ?

But $\lim_{n\to\infty}(1/2)^n = 0$, and so $\lim_{n\to\infty} c^n/n! = 0$ also. ■

* What is this saying?

Theorem 8.14 Let I be a neighborhood of the point x_0 and suppose that the function $f : I \to \mathbb{R}$ has derivatives of all orders. Suppose also that there are positive numbers r and M such that the interval $[x_0 - r, x_0 + r]$ is contained in I and that for every natural number n and every point x in $[x_0 - r, x_0 + r]$,

$$|f^{(n)}(x)| \leq M^n. \tag{8.13}$$

Then

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k \qquad \text{if } |x - x_0| \leq r. \tag{8.14}$$

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The Cauchy Integral Remainder Theorem (8.5)

Theorem 8.17 The Cauchy Integral Remainder Formula Let I be a neighborhood of the point x_0 and n be a natural number. Suppose that the function $f : I \to \mathbb{R}$ has $n+1$ derivatives and that $f^{(n+1)} : I \to \mathbb{R}$ is continuous. Then for each point x in I,

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k + \frac{1}{n!}\int_{x_0}^{x} f^{(n+1)}(t)(x-t)^n\, dt. \tag{8.24}$$

Proof

By the First Fundamental Theorem (Integrating Derivatives),

$$f(x) = f(x_0) + \int_{x_0}^{x} f'(t)\, dt. \tag{8.25}$$

Integrating by parts, we see that

$$\begin{aligned}\int_{x_0}^{x} f'(t)\, dt &= -\int_{x_0}^{x} f'(t)\frac{d}{dt}(x-t)\, dt \\ &= -f'(t)(x-t)\Big|_{t=x_0}^{t=x} + \int_{x_0}^{x} f''(t)(x-t)\, dt \\ &= f'(x_0)(x-x_0) + \int_{x_0}^{x} f''(t)(x-t)\, dt. \end{aligned} \tag{8.26}$$

From (8.25) and (8.26) we obtain (8.24) when $n = 1$. The general formula follows by induction. The inductive step depends on observing that if $1 \le k \le n-1$, then

$$\begin{aligned}\frac{1}{k!}\int_{x_0}^{x} f^{(k+1)}(t)(x-t)^k\, dt &= \frac{-1}{(k+1)!}\int_{x_0}^{x} f^{(k+1)}(t)\frac{d}{dt}[(x-t)^{k+1}]\, dt \\ &= \frac{1}{(k+1)!} f^{(k+1)}(x_0)(x-x_0)^{k+1} \\ &\quad + \frac{1}{(k+1)!}\int_{x_0}^{x} f^{(k+2)}(t)(x-t)^{k+1}\, dt. \end{aligned}$$

■

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#How does this proof compare to that of the Lagrange Remainder Theorem?

If I is a neighborhood of the point x_0 and the function $f: I \to \mathbb{R}$ is differentiable, then, by the Mean Value Theorem, for each point x in I, there is a point c strictly between x and x_0 such that

$$f(x) = f(x_0) + f'(c)(x - x_0). \tag{8.22}$$

If we further assume that the derivative $f': I \to \mathbb{R}$ is continuous, then, by the First Fundamental Theorem (Integrating Derivatives),

$$f(x) = f(x_0) + \int_{x_0}^{x} f'(t)\, dt. \tag{8.23}$$

The proof of the Lagrange Remainder Theorem was rooted in the Mean Value Theorem as expressed in (8.22). The proof of the following Cauchy Integral Remainder Theorem will exploit the First Fundamental Theorem (Integrating Derivatives) as expressed in (8.23).

#Why is the Cauchy Integral Remainder Theorem even more powerful than the LRT?

Theorem 8.8 The Lagrange Remainder Theorem Let I be a neighborhood of the point x_0 and let n be a nonnegative integer. Suppose that the function $f: I \to \mathbb{R}$ has $n + 1$ derivatives. Then for each point $x \neq x_0$ in I, there is a point c strictly between x and x_0 such that

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k + \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}. \tag{8.3}$$

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Sequences and Series of Numbers (9.1)

Theorem 9.1 The Monotone Convergence Theorem A monotone sequence of numbers converges if and only if it is bounded.

The Cauchy Convergence Criterion for Sequences

Definition A sequence of numbers $\{a_n\}$ is said to be a *Cauchy sequence* provided that for each positive number ϵ there is an index N such that

$$|a_n - a_m| < \epsilon \qquad \text{if } n \geq N \text{ and } m \geq N.$$

Proposition 9.2 Every convergent sequence is Cauchy.

Lemma 9.3 Every Cauchy sequence is bounded. $|a_n| \leq M$

Proof
Suppose that $\{a_n\}$ is a Cauchy sequence. For $\epsilon = 1$, we can choose an index N such that

$$|a_n - a_m| < 1 \qquad \text{if } n \geq N \text{ and } m \geq N.$$

In particular, we have

$$|a_n - a_N| < 1 \qquad \text{if } n \geq N.$$

But, setting

$$a_n = a_N + (a_n - a_N),$$

by the Triangle Inequality,

$$|a_n| = |a_N + (a_n - a_N)| \leq |a_N| + |a_n - a_N|.$$

Consequently, we see that

$$|a_n| \leq |a_N| + 1 \qquad \text{if } n \geq N.$$

?

Define $M = \max\{|a_N| + 1, |a_1|, |a_2|, \ldots, |a_{N-1}|\}$. Then

covers cases for $1 < n < N$

$$|a_n| \leq M \qquad \text{for every index } n. \qquad \blacksquare$$

Theorem 9.4 The Cauchy Convergence Criterion for Sequences A sequence of numbers converges if and only if it is a Cauchy sequence.

Proof

According to Proposition 9.2, every convergent sequence is a Cauchy sequence. The converse remains to be proven. Suppose that $\{a_n\}$ is a Cauchy sequence. The preceding lemma asserts that $\{a_n\}$ is bounded. Thus, by the Sequential Compactness Theorem, $\{a_n\}$ has a subsequence $\{a_{n_k}\}$ that converges to a number a.

We claim that the whole sequence $\{a_n\}$ converges to a. Indeed, let $\epsilon > 0$. We need to find an index N such that

$$|a_n - a| < \epsilon \qquad \text{if } n \geq N.$$

Since $\{a_n\}$ is a Cauchy sequence, we can choose an index N such that

$$|a_n - a_m| < \frac{\epsilon}{2} \qquad \text{if } n \geq N \text{ and } m \geq N. \tag{9.1}$$

On the other hand, since the subsequence $\{a_{n_k}\}$ converges to a, there is an index K such that

$$|a_{n_k} - a| < \frac{\epsilon}{2} \qquad \text{if } k \geq K. \tag{9.2}$$

Now choose any index k such that $k \geq K$ and $n_k \geq N$. Using the inequalities (9.1) and (9.2) together with the Triangle Inequality, it follows that if $n \geq N$, then

$$\begin{aligned} |a_n - a| &= |(a_n - a_{n_k}) + (a_{n_k} - a)| \\ &\leq |a_n - a_{n_k}| + |a_{n_k} - a| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

■

Convergence Tests for Series

Recall that for a sequence of numbers $\{a_k\}$ that is indexed by the natural numbers, we define

$$s_n = \sum_{k=1}^{n} a_k \qquad \text{for every index } n$$

and obtain a new sequence $\{s_n\}$. The sequence $\{s_n\}$ is called the *sequence of partial sums* for the series $\sum_{k=1}^{\infty} a_k$, and a_k is called the kth *term* of the series $\sum_{k=1}^{\infty} a_k$. We write

$$\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} \left[\sum_{k=1}^{n} a_k\right]$$

if the sequence $\{s_n\}$ converges. If the sequence $\{s_n\}$ does not converge, we say that the series $\sum_{k=1}^{\infty} a_k$ *diverges*.[1]

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* Helpful Facts

* Converse is not true: $\frac{1}{n}$

Proposition 9.5 Suppose that the series $\sum_{n=1}^{\infty} a_n$ converges. Then

"Divergence Test"

$$\lim_{n\to\infty} a_n = 0.$$

Proposition 9.6 For a number r such that $|r| < 1$,

$-1<r<1$

Geometric Series

$$\sum_{k=0}^{\infty} r^k = \frac{1}{1-r}.$$

Theorem 9.7 Suppose that $\{a_k\}$ is a sequence of nonnegative numbers. Then the series $\sum_{k=1}^{\infty} a_k$ converges if and only if the sequence of partial sums is bounded; that is, there is a positive number M such that

$$a_1 + \cdots + a_n \leq M \qquad \text{for every index } n.$$

* So far we are only dealing with positive series!

ex) $a_n = (-1)^n$

$\{a_n\} = \{a_0, a_1, a_2, \ldots\}$

$\{S_n\} = \{\underbrace{a_0}_{S_0}, \underbrace{a_0 + a_1}_{S_1}, \underbrace{a_0 + a_1 + a_2}_{S_2}, + \ldots\}$

$\{S_0, S_1, S_2 \ldots\}$

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Convergence Tests for Positive Series

1)

Corollary 9.11 The Integral Test Let $\{a_k\}$ be a sequence of nonnegative numbers and suppose that the function $f:[1,\infty) \to \mathbb{R}$ is continuous and monotonically decreasing and has the property that

$$f(k) = a_k \qquad \text{for every index } k.$$

Then the series $\sum_{k=1}^{\infty} a_k$ is convergent if and only if the sequence of integrals $\{\int_1^n f(x)\,dx\}$ is bounded.

ex) $\sum_{k=1}^{\infty} \frac{1}{k(k+1)} \to a_n$

$f(k) = \frac{1}{k(k+1)}$

$\sum_{k=1}^{\infty} a_n \Leftrightarrow \int_1^n \frac{1}{k(k+1)}\,dk$

$* \lim_{n\to\infty} \int_1^n \frac{1}{k(k+1)}\,dk$

2)

Corollary 9.13 The p-Test For a positive number p, the series

$$\sum_{k=1}^{\infty} \frac{1}{k^p}$$

converges if and only if $p > 1$.

3)

Corollary 9.8 The Comparison Test Suppose that $\{a_k\}$ and $\{b_k\}$ are sequences of numbers such that for index k,

$$0 \le a_k \le b_k.$$

i. The series $\sum_{k=1}^{\infty} a_k$ converges if the series $\sum_{k=1}^{\infty} b_k$ converges.

ii. The series $\sum_{k=1}^{\infty} b_k$ diverges if the series $\sum_{k=1}^{\infty} a_k$ diverges.

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New Study Resource!

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Convergence Tests for General (e.g., alternating) Series

1)

Theorem 9.15 The Alternating Series Test Suppose that $\{a_k\}$ is a monotonically decreasing sequence of nonnegative numbers that converges to 0. Then the series

$$\sum_{k=1}^{\infty}(-1)^{k+1}a_k$$

converges.

2)

Theorem 9.17 The Cauchy Convergence Criterion for Series The series $\sum_{k=1}^{\infty} a_k$ converges if and only if for each positive number ϵ there is an index N such that

$$|a_{n+1}+\cdots+a_{n+k}|<\epsilon \qquad \text{for all indices } n \geq N \text{ and all natural numbers } k.$$

Proof

?

Apply the Cauchy Convergence Criterion for Sequences to the sequence of partial sums. ■

* If $\sum_{k=1}^{\infty} a_k$ converges, then what do we know about: $\{a_1, a_1+a_2, a_1+a_2+a_3, \ldots\}$?

3)

Corollary 9.18 The Absolute Convergence Test An absolutely convergent series converges; that is, the series $\sum_{k=1}^{\infty} a_k$ converges if the series $\sum_{k=1}^{\infty} |a_k|$ converges.

4)

Theorem 9.20 For the series $\sum_{k=1}^{\infty} a_k$, suppose that there is a number r with $0 \leq r < 1$ and an index N such that

$$|a_{n+1}| \leq r|a_n| \qquad \text{for all indices } n \geq N. \tag{9.7}$$

Then the series $\sum_{k=1}^{\infty} a_k$ is absolutely convergent.

5)

Corollary 9.21 The Ratio Test for Series For the series $\sum_{k=1}^{\infty} a_k$, suppose that

$$\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|}=\ell.$$

i. If $\ell < 1$, the series $\sum_{n=1}^{\infty} a_n$ converges absolutely.
ii. If $\ell > 1$, the series $\sum_{n=1}^{\infty} a_n$ diverges.

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Pointwise Convergence of Sequences of Functions (9.2)

Chapter 3 was devoted to the study of sequences of numbers. In Section 9.1, we studied sequences of numbers and then series of numbers. We now turn to the study of sequences of *functions*.

$\{f_1, f_2, f_3, \dots f$

Definition Given a function $f : D \to \mathbb{R}$ and a sequence of functions $\{f_n : D \to \mathbb{R}\}$, we say that the sequence $\{f_n : D \to \mathbb{R}\}$ *converges pointwise* to $f : D \to \mathbb{R}$, or that $\{f_n\}$ *converges pointwise* on D to f, provided that for each point x in D,

$$\lim_{n\to\infty} f_n(x) = f(x).$$

Example 9.26 For each natural number n, define

$$f_n(x) = \sum_{k=0}^{n} \frac{x^k}{k!} \qquad \text{for } 0 \le x \le 1.$$

According to the Taylor series expansion formula (8.16), the sequence of functions $\{f_n\}$ converges pointwise on the interval $[0, 1]$ to the function e^x. In infinite series notation, this is expressed as

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} \qquad \text{for } 0 \le x \le 1.$$

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Example 9.22 For each natural number n, define

$$f_n(x) = x^n \qquad \text{for } 0 \le x \le 1.$$

Since $\{f_n(1)\}$ is a constant sequence whose constant value is 1.

$$\lim_{n\to\infty} f_n(1) = 1.$$

On the other hand,

$$\lim_{n\to\infty} x^n = 0 \qquad \text{if } 0 \le x < 1.$$

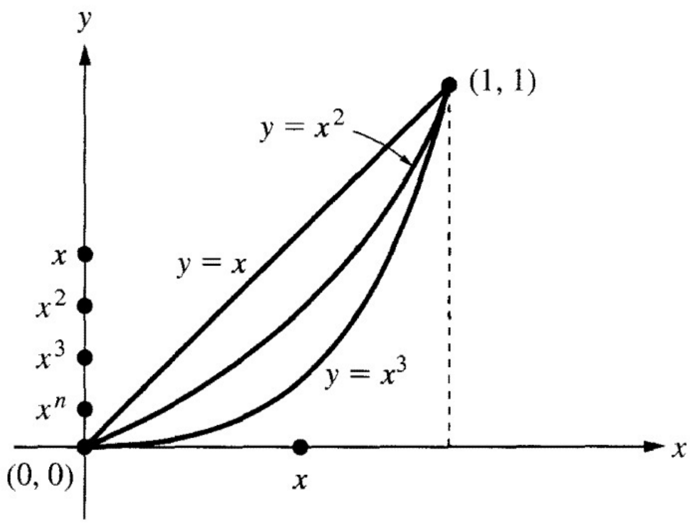


FIGURE 9.2 $\lim_{n\to\infty} x^n = 0$ if $0 \le x < 1$; $\lim_{n\to\infty} 1^n = 0$.

Thus, the sequence of functions $\{f_n\}$ converges pointwise on $[0,\ 1]$ to the function f defined by

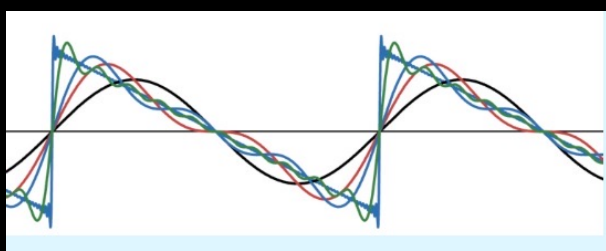
$$f(x) = \begin{cases} 1 & \text{if } x = 1 \\ 0 & \text{if } 0 \le x < 1. \end{cases}$$

Observe that this is an example of a sequence of continuous functions that converges pointwise to a discontinuous function. ■

*Another Example:

$$f_n(x) = \sum_{n=1}^{\infty} \frac{1}{n} \sin(nx)$$

"Sawtooth function"



borrowed from Dr. Alex Kontorov

* (see more here).

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* Other interesting Examples

(a)

Example 9.23 For each natural number n, define

$$f_n(x) = e^{-nx^2} \qquad \text{for all } x.$$

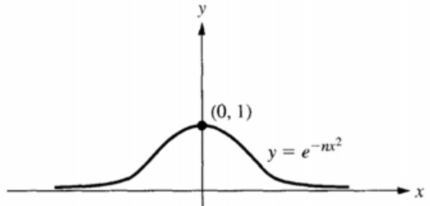

FIGURE 9.3 $\lim_{n\to\infty} e^{-nx^2} = 0$ if $x \neq 0$; $\lim_{n\to\infty} e^{-n[0]^2} = 1$.

Thus, the sequence of functions $\{f_n\}$ converges pointwise on $\mathbb{R}$ to the function f defined by

$$f(x) = \begin{cases} 0 & \text{if } x \neq 0 \\ 1 & \text{if } x = 0. \end{cases}$$

Observe that this is an example of a sequence of functions, each of which is differentiable on $\mathbb{R}$, that converges pointwise on $\mathbb{R}$ to a function that is not differentiable at $x = 0$. ■

(b)

Example 9.24 A number x of the form $x = k/2^n$, for an integer k and a natural number n, is called a *dyadic rational*. For each natural number n and each number x in the interval $[0, 1]$, define

$$f_n(x) = \begin{cases} 1 & \text{if } x = k/2^n \text{ for some natural number } k \\ 0 & \text{otherwise.} \end{cases}$$

Thus, the sequence of functions $\{f_n\}$ converges pointwise on the interval $[0, 1]$ to the function f defined by

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is a dyadic rational} \\ 0 & \text{otherwise.} \end{cases}$$

This is an example of a sequence of integrable functions that converges pointwise on a closed bounded interval to a function that is not integrable (Exercise 5). ■

Let $k=1$: $f_1 = \begin{cases} 1, & x = \frac{1}{2} \\ 0, & \text{otherwise} \end{cases}$, $f_2 = \begin{cases} 1, & x = \frac{1}{4} \\ 0, & \text{otherwise} \end{cases}$, ...

$\hookrightarrow \int_0^1 f_1 = 0$ $\qquad$ $\hookrightarrow \int_0^1 f_2 = 0$, ...

But, $f(x)$ is not integrable! $\rightarrow U(f, P_n) = 1$ for every partition (dyadics are dense)

$L(f, P_n) = 0$ for every partition

so $\lim_{n\to\infty} U(f,P_n) - L(f,P_n) \neq 0$

Example 9.25 For a natural number n, define $f_n : [0,\ 1] \to \mathbb{R}$ to be the function such that $f_n(0) = f(2/n) = f_n(1) = 0$, $f_n(1/n) = n$, and f_n is linear on the intervals $[0,\ 1/n]$, $[1/n,\ 2/n]$, and $[2/n,\ 0]$.

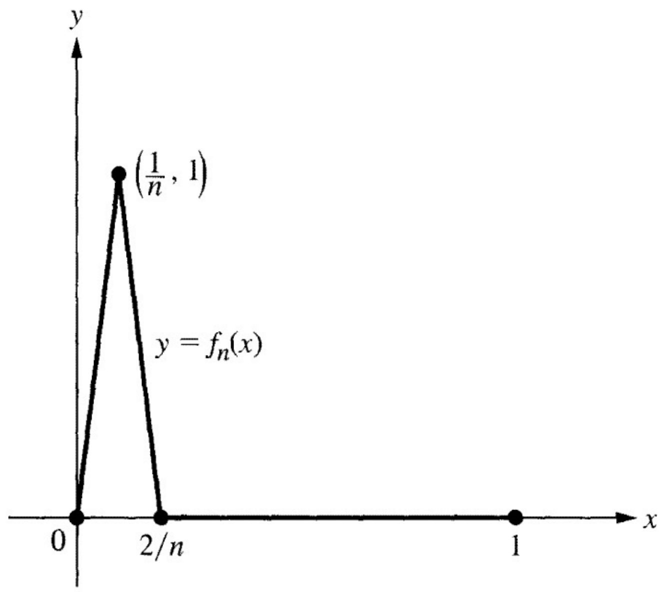

FIGURE 9.4 $f_n(x) = 0$ if $2/n \leq x \leq 1$; $\int_0^1 f_n = 1$.

Thus, the sequence of functions $\{f_n\}$ converges pointwise on the interval $[0,\ 1]$ to 0 (by this we mean to the function that is identically equal to 0 on $[0,\ 1]$). Observe that $\int_0^1 f = 0$, while for each index n, $\int_0^1 f_n = 1$. ■

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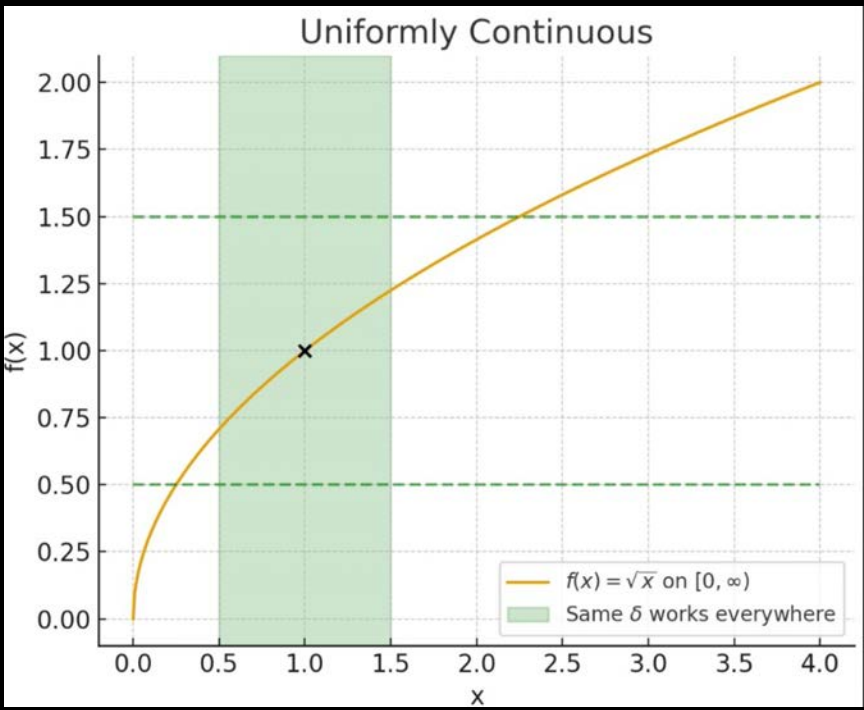
Uniform Convergence of Sequences of Function (9.3)

Recall:



Definition Given a function $f : D \to \mathbb{R}$ and a sequence of functions $\{f_n : D \to \mathbb{R}\}$, the sequence $\{f_n : D \to \mathbb{R}\}$ is said to *converge uniformly* to $f : D \to \mathbb{R}$, or $\{f_n\}$ is said to *converge uniformly* on D to f, provided that for each positive number ϵ there is an index N such that

$$|f(x) - f_n(x)| < \epsilon \qquad \text{for all indices } n \geq N \text{ and all points } x \text{ in } D. \tag{9.9}$$

In terms of graphs, the sequence $\{f_n : D \to \mathbb{R}\}$ converges uniformly to $f : D \to \mathbb{R}$ if for each positive number ϵ there is a natural number N such that if $n \geq N$, the graph of the function $f_n : D \to \mathbb{R}$ lies between the graphs of the functions $f + \epsilon : D \to \mathbb{R}$ and $f - \epsilon : D \to \mathbb{R}$.

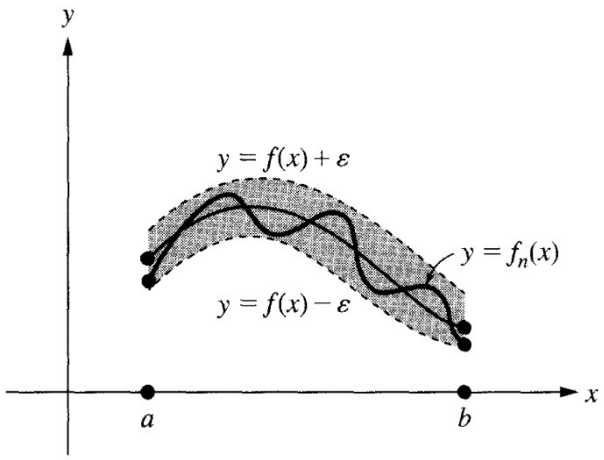


FIGURE 9.5 $f(x) - \epsilon < f_n(x) < f(x) + \epsilon$ if $a \leq x \leq b$ and $n \geq N$.

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does **not** converge uniformly

non-ex) $f_n(x) = x^n$, $0 \le x \le 1$, $n \in \mathbb{N}$

↳ converges to $f(x) = \begin{cases} 1, & x = 1 \\ 0, & 0 \le x < 1 \end{cases}$

(from previous section)

ex) See Homework Problem # 57

For each natural number n and each number x in $[0, 1]$, define

$$f_n(x) = \frac{x}{nx + 1}$$

*One last result from this section...

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Definition The sequence of functions $\{f_n : D \to \mathbb{R}\}$ is said to be *uniformly Cauchy*, or $\{f_n\}$ is said to be *uniformly Cauchy* on D, provided that for each positive number ϵ there is an index N such that

$$|f_{n+k}(x) - f_n(x)| < \epsilon \tag{9.11}$$

for every index $n \geq N$, every natural number k, and every point x in D.

Theorem 9.29 The Weierstrass Uniform Convergence Criterion The sequence of functions $\{f_n : D \to \mathbb{R}\}$ converges uniformly to a function $f : D \to \mathbb{R}$ if and only if the sequence $\{f_n : D \to \mathbb{R}\}$ is uniformly Cauchy.

? Why does this feel familiar?

↳ *A sequence converges uniformly iff it is Cauchy.

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Example 9.30 For each natural number n and each number x with $|x| \leq 1$, define

$$f_n(x) = \sum_{k=1}^{n} \frac{x^k}{k2^k}.$$

Observe, using the Triangle Inequality and the Geometric Sum Formula, that for each pair of natural numbers n and k and each number x with $|x| \leq 1$,

$$\begin{aligned} |f_{n+k}(x) - f_n(x)| &\leq \frac{|x|^{n+1}}{(n+1)2^{n+1}} + \cdots + \frac{|x|^{n+k}}{(n+k)2^{n+k}} \quad ? \\ &\leq \frac{1}{2^{n+1}} + \cdots + \frac{1}{2^{n+k}} \quad ? \\ &\leq \frac{1}{2^n}. \end{aligned} \tag{9.14}$$

But $\lim_{n\to\infty}(1/2)^n = 0$, and this, together with the inequality (9.14), implies that the sequence $\{f_n : [-1, 1] \to \mathbb{R}\}$ is uniformly Cauchy. According to the Weierstrass Uniform Convergence Criterion, there is a function $f : [-1, 1] \to \mathbb{R}$ to which the sequence $\{f_n : [-1, 1] \to \mathbb{R}\}$ converges uniformly. ■

$$\frac{|x|^{n+1}}{(n+1)2^{n+1}} + \cdots + \frac{|x|^{n+k}}{(n+k)2^{n+k}} \leq \frac{1}{(n+1)2^{n+1}} + \cdots + \frac{1}{(n+k)2^{n+k}} \leq \frac{1}{2^{n+1}} + \cdots + \frac{1}{2^{n+k}}$$

$$= \sum_{k=0}^{n+k} \left(\frac{1}{2}\right)^k - \sum_{k=0}^{n} \left(\frac{1}{2}\right)^k$$

$$= \frac{1-\left(\frac{1}{2}\right)^{n+k+1}}{\frac{1}{2}} - \frac{1-\left(\frac{1}{2}\right)^{n+1}}{\frac{1}{2}}$$

$$= 2 - \left(\frac{1}{2}\right)^{n+k} - 2 + \left(\frac{1}{2}\right)^n$$

$$= \left(\frac{1}{2}\right)^n - \left(\frac{1}{2}\right)^{n+k}$$

$$= \left(\frac{1}{2}\right)^n \left[1-\left(\frac{1}{2}\right)^k\right] \leq \left(\frac{1}{2}\right)^n (1)$$

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The Uniform Limit Of A Function (9.4)

*Recall:

$f_n \xrightarrow{pw} f$: For all $x \in D$ and for all $\varepsilon > 0$, there exists N s.t. $|f_n(x) - f(x)| < \varepsilon$ for $n \geq N$.

$f_n \xrightarrow{u} f$: For all $\varepsilon > 0$, there exists N s.t. for all $x \in D$, $|f_n(x) - f(x)| < \varepsilon$ for $n \geq N$.

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Theorem 9.31 Suppose that $\{f_n : D \to \mathbb{R}\}$ is a sequence of continuous functions that converges uniformly to the function $f : D \to \mathbb{R}$. Then the limit function $f : D \to \mathbb{R}$ also is continuous.

Theorem 9.32 Suppose that $\{f_n : [a, b] \to \mathbb{R}\}$ is a sequence of integrable functions that converges uniformly to the function $f : [a, b] \to \mathbb{R}$. Then the limit function $f : [a, b] \to \mathbb{R}$ also is integrable. Moreover,

$$\lim_{n\to\infty} \left[\int_a^b f_n \right] = \int_a^b f.$$

i.e., each derivative is continuous!

Theorem 9.33 Let I be an open interval. Suppose that $\{f_n : I \to \mathbb{R}\}$ is a sequence of continuously differentiable functions that has the following two properties:

i. The sequence $\{f_n\}$ converges pointwise on I to the function f, and
ii. The derived sequence $\{f_n'\}$ converges uniformly on I to the function g.

Then the function $f : I \to \mathbb{R}$ is continuously differentiable, and

$$f'(x) = g(x) \qquad \text{for all } x \text{ in } I.$$

* More general version of Thm. 9.33

Theorem 9.34 Let I be an open interval. Suppose that $\{f_n : I \to \mathbb{R}\}$ is a sequence of continuously differentiable functions that has the following two properties:

i. The sequence $\{f_n\}$ converges pointwise on I to the function f, and
ii. The derived sequence $\{f_n'\}$ is uniformly Cauchy on I.

Then the function $f : I \to \mathbb{R}$ is continuously differentiable, and for each x in I,

$$\lim_{n\to\infty} f_n'(x) = f'(x).$$

* Wow. That is a lot of info! Let's organize.

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Continuous — Converges Un.

$f_1, f_2, f_3, \ldots, f_n$ $\quad \{f_n\} \to f \quad \longrightarrow f$ continuous

---

Integrable — Converges Un

$f_1, f_2, f_3, \ldots, f_n$ $\quad \{f_n\} \to f \quad \longrightarrow f$ integrable

---

Continuous — Pointwise — Un. Cont.

$f_1, f_2, f_3, \ldots, f_n$ $\quad \{f_n\} \xrightarrow{PW} f \quad \{f_n'\} \xrightarrow{U} g \quad \to$ $\begin{cases} f \text{ continuous} \\ f' \text{ continuous} \\ f' = g \end{cases}$

$f_1', f_2', f_3', \ldots, f_n'$

Continuous — Pointwise — Un. Cauchy

$f_1, f_2, f_3, \ldots, f_n$ $\quad \{f_n\} \xrightarrow{PW} f \quad \{f_n'\}$ uniformly Cauchy $\to$ $\begin{cases} f \text{ continuous} \\ f' \text{ continuous} \\ \lim_{n \to \infty} f_n' = f' \end{cases}$

$f_1', f_2', f_3', \ldots, f_n'$

don't need to know what is converges to ahead of time! Just look at f'

* Challenge Examples:

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1. For each natural number n and each number x in $(-1, 1)$, define

$$f_n(x) = \sqrt{x^2 + \frac{1}{n}}$$

and define $f(x) = |x|$. Prove that the sequence $\{f_n\}$ converges uniformly on the open interval $(-1, 1)$ to the function f. Check that each function f_n is continuously differentiable, whereas the limit function f is not differentiable at $x = 0$. Does this contradict Theorem 9.33?

2. For each natural number n and each number x in $[0, 1]$, define

$$f_n(x) = nxe^{-nx^2}.$$

Prove that the sequence $\{f_n\}$ converges pointwise on the interval $[0, 1]$ to the constant function 0, but that the sequence of integrals $\{\int_0^1 f_n\}$ does not converge to 0. Does this contradict Theorem 9.32?

3. Prove that if $\{f_n : \mathbb{R} \to \mathbb{R}\}$ is a sequence of continuously differentiable functions such that the sequence of derivatives $\{f_n' : \mathbb{R} \to \mathbb{R}\}$ is uniformly convergent and the sequence $\{f_n(0)\}$ is also convergent, then $\{f_n : \mathbb{R} \to \mathbb{R}\}$ is pointwise convergent. Is the assumption that the sequence $\{f_n(0)\}$ converges necessary?

Power Series (9.5)

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*Aim - Let's expand on the idea of a Taylor Series!

Definition Given a sequence of real numbers $\{c_k\}$ indexed by the nonnegative integers, we define the *domain of convergence* of the series $\sum_{k=0}^{\infty} c_k x^k$ to be the set of all numbers x such that the series $\sum_{k=0}^{\infty} c_k x^k$ converges. Denote the domain of convergence by D. We then define a function $f : D \to \mathbb{R}$ by

Also known as "radius of convergence"

$$f(x) = \lim_{n\to\infty} \left[\sum_{k=0}^{n} c_k x^k\right] = \sum_{k=0}^{\infty} c_k x^k \qquad \text{for all } x \text{ in } D. \tag{9.28}$$

We refer to (9.28) as a *power series expansion* and call the set D the *domain of convergence of the expansion.*

* Key Result

Proposition 9.40 Suppose that the nonzero number x_0 is in the domain of convergence of the power series $\sum_{k=0}^{\infty} c_k x^k$. Let r be any positive number less than $|x_0|$. Then the interval $[-r, r]$ is in the domain of convergence of the power series $\sum_{k=0}^{\infty} c_k x^k$ and also in the domain of convergence of the derived power series $\sum_{k=1}^{\infty} k c_k x^{k-1}$. Moreover, each of the power series

$$\sum_{k=0}^{\infty} c_k x^k \quad \text{and} \quad \sum_{k=1}^{\infty} k c_k x^{k-1}$$

converges uniformly on the interval $[-r, r]$.

Final Exam

Review

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1) Prove, by contradiction, the set $\mathbb{Q}$ is dense in $\mathbb{R}$.

Dense - $\mathbb{Q}$ is dense provided that if $a, b \in \mathbb{R}$ with $a<b$, there exists some $\frac{m}{n}$ such that $a < \frac{m}{n} < b$, with $m, n \in \mathbb{Z}$ and $n \neq 0$.

Proof - Suppose $\mathbb{Q}$ is not dense. Then there exists $a, b \in \mathbb{R}$ with $a<b$ such that for all $m, n \in \mathbb{Z}$, with $n \neq 0$, $\frac{m}{n} \notin (a,b)$. By the AP, we can find $n \in \mathbb{N}$, so $n \in \mathbb{Z}$, such that $0 < \frac{1}{n} < b-a$. Moreover, note that there is an integer m such that $na \leq m < na+1$. In turn, we have $a \leq \frac{m}{n} < a + \frac{1}{n} < a + [b-a] = b$, so $a \leq \frac{m}{n} < b$. Note that in the case that $a \in \mathbb{Z}$, then so is $na+1$, so we can say there is $m \in \mathbb{Z}$ with $na \leq m \leq na+1$. Thus, in every case $\frac{m}{n} \in (a,b)$, a contradiction. $\blacksquare$

*Remark: Could use $k \in [na, na+1]$ to cover both cases simultaneously!

*Scrap work →

Scrap work

$a < \frac{m}{n} < b$

$na < m < nb$ we'll make sure $n \in \mathbb{N}$, so $n > 0$

want: $na < m < na + 1 < nb$

$a < \frac{m}{n} < a + \frac{1}{n} < b$

$0 < \frac{m}{n} - a < \frac{1}{n} < b - a$ invoke AP

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2) Let $g(x) = x + \frac{1}{x}$, for $x \geq 1$.
Prove g is uniformly continuous.

Uniformly Continuous - For every $\varepsilon > 0$ there exists $\delta > 0$ such that for all $a, b \in Dom(g)$, if $|a-b| < \delta$, then $|f(a) - f(b)| < \varepsilon$.

Proof:
Let $a, b \in Dom(g)$ with $a, b \geq 1$. Then observe that for every $\varepsilon > 0$, if $\delta = \frac{\varepsilon}{2}$ and $|a-b| < \delta$, we have the following:

$$
\begin{aligned}
|f(a) - f(b)| &= \left|a + \frac{1}{a} - \left\{b + \frac{1}{b}\right\}\right| \\
&\leq |a-b| + \left|\frac{1}{a} - \frac{1}{b}\right| \\
&< \frac{\varepsilon}{2} + \left|\frac{b-a}{ab}\right| \\
&\leq \frac{\varepsilon}{2} + (b-a) \\
&= \frac{\varepsilon}{2} + |a-b| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon.
\end{aligned}
$$

Thus, g is uniformly continuous for $x \geq 1$. ☒

* Scrap work →

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Scrap work

$$\left|a+\frac{1}{a}-\left(b+\frac{1}{b}\right)\right|<\varepsilon$$

$$\left|a-b+\frac{1}{a}-\frac{1}{b}\right|<\varepsilon$$

$$|a-b|+\left|\frac{1}{a}-\frac{1}{b}\right|<\varepsilon$$

$$\delta+\left|\frac{1}{a}-\frac{1}{b}\right|<\varepsilon$$

$$\delta+\left|\frac{b-a}{ab}\right|<\varepsilon$$

$$\delta+\frac{|b-a|}{ab}<\varepsilon$$

$$\delta+|b-a|<\varepsilon$$

$$\delta+|a-b|<\varepsilon$$

$$2\delta<\varepsilon$$

$$\delta<\frac{\varepsilon}{2}$$

$a\geq 1$

$b\geq 1$

$ab\geq 1$

$\frac{1}{ab}\leq 1$

$\frac{|b-a|}{ab}\leq |b-a|$

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3) Prove or Disprove

(a) The set $\{\frac{1}{n} \mid n \in \mathbb{N}\}$ is dense in $[0,1]$.

*Hint: What happens if you pick successive numbers of the form $\frac{1}{n}$ that fall within $[0,1]$?

False! Let $a = \frac{1}{3}$ and $b = \frac{1}{2}$, so $a < b$ and $a, b \in [0,1]$. Then observe that there does not exist $n \in \mathbb{N}$ such that $\frac{1}{3} < \frac{1}{n} < \frac{1}{2}$ since there does not exist a natural number between 2 and 3. Thus, not all numbers between 0 and 1 can be captured by our set.

(b) The infinite series $\sum_{n=1}^{\infty} \frac{\sin(nx)}{n^2}$ converges ~~uniformly~~ pointwise on the interval $[0, \infty)$.

*Hint: Compare $\frac{\sin(nx)}{n^2}$ to another sequence.

Proof: Observe that $\left|\frac{\sin(nx)}{n^2}\right| \le \frac{1}{n^2}$, for all n and for all $x \in [0,\infty)$. Moreover, $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by p-test. Thus, by the absolute Comparison Test, $\sum_{n=1}^{\infty} \left|\frac{\sin(nx)}{n^2}\right|$ converges. More specifically, $f_n = \sum_{n=1}^{\infty} \frac{\sin(nx)}{n^2}$ converges pointwise to 0. ∎

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(c) If $\lim_{n\to\infty} [a_{n+1} - a_n] = 0$, then $\lim_{n\to\infty} a_n$ exists.

*Hint: Make a_n a sequence of partial sums, specifically using a series you know diverges.

False! Consider the counterexample $a_n = \sum_{k=1}^{n} \frac{1}{k}$. Then note that $\lim_{n\to\infty} a_n$ does not exist (i.e., $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges by p-test).

However, $a_{n+1} - a_n = \sum_{k=1}^{n+1} \frac{1}{k} - \sum_{k=1}^{n} \frac{1}{k} = \frac{1}{n+1}$, so

$\lim_{n\to\infty} a_{n+1} - a_n = 0$.

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4) Prove the following equation has exactly one solution.
*Note you must prove existence and uniqueness.

$$e^x + \cos(x) + x = 0$$

Proof:

Let $f(x) = e^x + \cos(x) + x$. Observe that

$f(0) = e^{(0)} + \cos(0) + (0) = 1 + 1 + 0 = 2$, and

$f(-\pi) = e^{(-\pi)} + \cos(-\pi) + (-\pi) = \frac{1}{e^{\pi}} - 1 - \pi < 0$

(note $\frac{1}{e^{\pi}} < 1$). The by the IVT, there exists

$c \in (-\pi, 0)$ such that $f(c) = 0$.

Suppose there exists another point ℓ such that $f(\ell) = 0$. Then by Rolle's Theorem (with $a = c$ and $b = \ell$), there exists $i \in (c, \ell)$ such that $f'(i) = \frac{f(\ell) - f(c)}{\ell - c} = 0$.

Note $f'(x) = e^x - \sin(x) + 1$, so we have the following:

$$f'(i) = e^i - \sin(i) + 1 = 0$$
$$e^i = \sin(i) - 1.$$

But $e^i > 0$ and $\sin(i) - 1 \leq 1 - 1 = 0$, so there is no solution the above equation!

Therefore, c must be unique. $\blacksquare$

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5) Let I be a neighborhood of a real number x_0, and let $n \in \mathbb{N}$.
Suppose $f: I \to \mathbb{R}$ has $n+1$ derivatives, and that $f^{(n+1)}: I \to \mathbb{R}$ is continuous.
Assume $f^{(k)}(x_0) = 0$ for all k, with $1 \le k \le n$.
Suppose $f^{(n+1)}(x_0) < 0$ and n is odd.
Prove x_0 is a local maximizer of f.

Proof:

Observe that the Lagrange Remainder Theorem,
$$f(x) = \sum_{k=0}^{n} \frac{f^{k}(x_0)(x-x_0)^k}{k!} + \frac{f^{n+1}(c)(x-x_0)^{n+1}}{(n+1)!}, \text{ for}$$
c between x and x_0.
But this reduces to $f(x) = f(x_0) + \dfrac{f^{n+1}(c)(x-x_0)^{n+1}}{(n+1)!}$.

Moreover, since n is odd, $n+1$ is even, so $(x-x_0)^{n+1} \ge 0$.
In particular, since $f^{(n+1)}$ is continuous (differentiability → continuity), and $f^{n+1}(x_0) < 0$, we can choose a neighborhood of x_0 such that $f^{n+1}(x) < 0$ for all x in this neighborhood, namely c.
Thus, $\dfrac{f^{n+1}(c)(x-x_0)^{n+1}}{(n+1)!} < 0$, and in turn, $f(x) - f(x_0) < 0$ or $f(x) < f(x_0)$. Therefore, x_0 is a local maximizer of f. ∎

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6) Prove that the following function is continuous over $\mathbb{R}$.

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{x^{2n} + n^3}$$

* i.e., Show that $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{x^{2n}+n^3}$ is continuous (or the limiting function $f(x)$ above is continuous).

Note that $\{f_n\}$ is a sequence of continuous functions since, for all n, f_n is the sum of quotients of polynomials, all of which are defined for all $x \in \mathbb{R}$. Moreover, note $\{f_n\}$ converges uniformly since this sequence is uniformly cauchy, as proven on the next page:

First, $\sum_{k=1}^{\infty} \frac{1}{k^3}$ converges by the p-test. So,

for $\varepsilon > 0$, choose N s.t. if $n \geq N$, $\left| \sum_{k=n}^{\infty} \frac{1}{k^3} \right| < \varepsilon$. (Cauchy convergence of $\sum_{k=1}^{\infty} \frac{1}{k^3}$).

Now observe:

$$|f_{n+k} - f_n| = \left| \sum_{k=1}^{n+k} \frac{1}{x^{2k}+k^3} - \sum_{k=1}^{n} \frac{1}{x^{2k}+k^3} \right| = \left| \sum_{k=n+1}^{n+k} \frac{1}{x^{2k}+k^3} \right| < \sum_{k=n+1}^{n+k} \frac{1}{n^3} < \varepsilon, \text{ \underline{for all } x}.$$

That is, f_n uniformly Cauchy, and hence, uniformly Cauchy by the Weierstrass Uniform Convergence Criterion.

Thus, $\{f_n\}$ is a sequence of continuous functions that converge uniformly to $f(x)$, so $f(x)$ is also continuous (see note below). ∎

* Note: This is the only fact needed from Section 9.4, which is **not** on the final exam!

* That's a lot of vocab! Let's keep some examples straight...

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Continuous

Example 3.18 Define $f(x) = x^3$ for all x. We claim that the function $f : \mathbb{R} \to \mathbb{R}$ satisfies the ϵ-δ criterion at the point $x_0 = 2$. Let $\epsilon > 0$. We must find a $\delta > 0$ such that

$$|x^3 - 8| < \epsilon \qquad \text{if } |x - 2| < \delta. \tag{3.14}$$

But observe that the difference of cubes formula and the Triangle Inequality imply that for all x,

$$|x^3 - 8| = |(x^2 + 2x + 4)(x - 2)| \leq [|x|^2 + 2|x| + 4]|x - 2|.$$

However,

$$|x|^2 + 2|x| + 4 \leq 19 \qquad \text{if } 1 < x < 3$$

so that

$$|x^3 - 8| \leq 19|x - 2| \qquad \text{if } 1 < x < 3. \tag{3.15}$$

Define

$$\delta = \min\{1, \epsilon/19\}. \tag{3.16}$$

If $|x - 2| < \delta$, then $1 < x < 3$ and $19|x - 2| < \epsilon$, so from (3.15) it follows that $|x^3 - 8| < \epsilon$. Therefore, for $\epsilon > 0$, if we define $\delta > 0$ by (3.16), then condition (3.14) holds. ∎

Uniformly Continuous

Example 3.21 Define $f(x) = x^3$ for x in $[0, 20]$. Then the function $f : [0, 20] \to \mathbb{R}$ is uniformly continuous. To see this, observe that for all u and v in $[0, 20]$,

$$|f(u) - f(v)| = |u^2 + uv + v^2||u - v| \leq 1200|u - v|.$$

Hence, for $\epsilon > 0$, if we define $\delta = \epsilon/1200$, then (3.21) holds. ∎

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Convergent

Proposition 2.6 The sequence $\{1/n\}$ converges to 0; that is,

$$\lim_{n \to \infty} 1/n = 0.$$

Proof

Let $\epsilon > 0$. We need to find an index N such that

$$\left| \frac{1}{n} - 0 \right| < \epsilon \qquad \text{for all indices } n \geq N; \tag{2.2}$$

that is, $1/n < \epsilon$ if $n \geq N$. But by the Archimedean Property of $\mathbb{R}$, we can select an index N such that $1/N < \epsilon$, and hence

$$\frac{1}{n} \leq \frac{1}{N} < \epsilon \qquad \text{for all indices } n \geq N.$$

Thus, the required inequality (2.2) holds for this choice of N. ∎

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* Now for sequences of functions...

Uniformly Convergent

57) a) Let $x \in [0,1]$. If $x = 0$ then $f_n(0) = 0$, for all $n \in \mathbb{N}$. For $x \in (0,1]$, we have $f_n(x) = \dfrac{x}{nx+1} = \dfrac{1}{n + \frac{1}{x}} < \dfrac{1}{n}$. But from previous work in the semester, we know that the sequence whose terms are all 0 and the sequence $\{\frac{1}{n}\}$ both converge to 0. So the sequence of functions $f_n(x)$ converges pointwise to $f(x) = 0$ for all x on $[0,1]$.

b) **Proof:** To prove uniform convergence, note that if $n \in \mathbb{N}$ and $x \in [0,1]$, then

$$|f_n(x) - f(x)| = \left|\frac{x}{nx+1} - 0\right| = \frac{x}{nx+1}$$

But using our work above and the Archimedean Property, we can choose $N \in \mathbb{N}$ such that $1/N < \epsilon$. In turn, if $n \geq N$, then $|f_n(x) - f(x)| < \epsilon$, in *all* cases of $x \in [0,1]$. Thus $\{f_n\}$ converges to f uniformly on $[0,1]$. ∎

Pointwise Convergent

Example 9.22 For each natural number n, define

$$f_n(x) = x^n \qquad \text{for } 0 \leq x \leq 1.$$

Since $\{f_n(1)\}$ is a constant sequence whose constant value is 1.

$$\lim_{n\to\infty} f_n(1) = 1.$$

On the other hand,

$$\lim_{n\to\infty} x^n = 0 \qquad \text{if } 0 \leq x < 1.$$

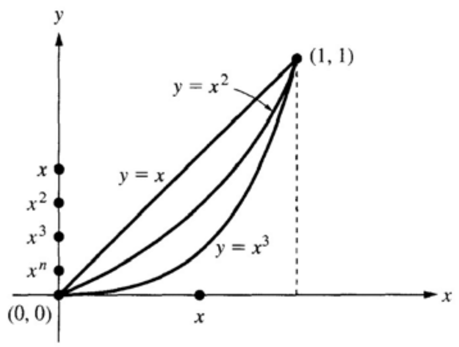

FIGURE 9.2 $\lim_{n\to\infty} x^n = 0$ if $0 \leq x < 1$; $\lim_{n\to\infty} 1^n = 0$.

Thus, the sequence of functions $\{f_n\}$ converges pointwise on $[0, 1]$ to the function f defined by

$$f(x) = \begin{cases} 1 & \text{if } x = 1 \\ 0 & \text{if } 0 \leq x < 1. \end{cases}$$

Observe that this is an example of a sequence of continuous functions that converges pointwise to a discontinuous function. ∎

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53) **Proof:** Let $\epsilon > 0$ be given, choose $N > \frac{2}{\epsilon}$ by the AP property. Then, for all $m, n \geq N$, we have:

$$|a_n - a_m| = \left|\left(\frac{1}{n} + 4\right) - \left(\frac{1}{m} + 4\right)\right| = \left|\frac{1}{n} - \frac{1}{m}\right| \leq \left|\frac{1}{n}\right| + \left|\frac{1}{m}\right| \leq \frac{1}{N} + \frac{1}{N} = \frac{2}{N} < \epsilon$$

Thus, the sequence is Cauchy by definition.

$\hookrightarrow \{\frac{1}{n} + 4\}$

Uniformly Cauchy

Example 9.30 For each natural number n and each number x with $|x| \leq 1$, define

$$f_n(x) = \sum_{k=1}^{n} \frac{x^k}{k2^k}.$$

Observe, using the Triangle Inequality and the Geometric Sum Formula, that for each pair of natural numbers n and k and each number x with $|x| \leq 1$,

$$\begin{aligned} |f_{n+k}(x) - f_n(x)| &\leq \frac{|x|^{n+1}}{(n+1)2^{n+1}} + \cdots + \frac{|x|^{n+k}}{(n+k)2^{n+k}} \\ &\leq \frac{1}{2^{n+1}} + \cdots + \frac{1}{2^{n+k}} \\ &\leq \frac{1}{2^n}. \end{aligned} \tag{9.14}$$

But $\lim_{n\to\infty}(1/2)^n = 0$, and this, together with the inequality (9.14), implies that the sequence $\{f_n : [-1, 1] \to \mathbb{R}\}$ is uniformly Cauchy. According to the Weierstrass Uniform Convergence Criterion, there is a function $f : [-1, 1] \to \mathbb{R}$ to which the sequence $\{f_n : [-1, 1] \to \mathbb{R}\}$ converges uniformly. ■

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Discussion Worksheets

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Math410: Discussion WS #1

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We have proven that the sequence $\{1/n\}$ converges to 0 and that it does not converge to any other number. Use this to prove that none of the following assertions is equivalent to the definition of convergence of a sequence $\{a_n\}$ to the number a.

1) For some $\epsilon > 0$ there is an index N such that

$$|a_n - a| < \epsilon \quad \text{for all indices } n \geq N.$$

* keep ε generic, followed by generically choosing N

Choose some $\varepsilon > 0$. By this new definition, there exists N such that $|\frac{1}{n}| < \varepsilon$ for all $n \geq N$, so $\frac{1}{n}$ converges to 0 as expected.

Now let $\varepsilon^* = \varepsilon - \frac{\varepsilon}{2}$ (observe $\varepsilon - \frac{\varepsilon}{2} > 0$).

Note that by our old definition of convergence, there exists an N such that $|\frac{1}{n}| < \varepsilon - \frac{\varepsilon}{2}$, or $|\frac{1}{n}| < \varepsilon^*$ for all $n \geq N$. That is, $|\frac{1}{n}| + \frac{\varepsilon}{2} = |\frac{1}{n} + \frac{\varepsilon}{2}| < \varepsilon$. However, by our new definition, this implies $\frac{1}{n}$ converges also to $-\frac{\varepsilon}{2}$.

2) For each $\epsilon > 0$ and each index N,

* Freely choose ε and N

$$|a_n - a| < \epsilon \quad \text{for all indices } n \geq N.$$

Let $\varepsilon = 1$ and $N = 1$. Then it is not the case that $|\frac{1}{n}| < 1$ for $n \geq N$, in particular, for $n = 1$.

Yet by our old definition, it should be that we can find an index N such that $|\frac{1}{n}| < 1$ for all $n \geq N$.

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Math410: Discussion WS #1

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3) There is an index N such that for every number $\epsilon > 0$,

$$|a_n - a| < \epsilon \quad \text{for all indices } n \geq N.$$

* keep N generic, followed by freely choose ε

Using this new definition of convergence, there is an index N such that for $\varepsilon = \frac{1}{N}$ (observe $\frac{1}{N} > 0$), $\left|\frac{1}{n}\right| < \frac{1}{N}$ for all $n \geq N$. However, observe that $\frac{1}{n} = \frac{1}{N}$ for $n = N$, that is, it is not true that $\left|\frac{1}{n}\right| < \frac{1}{N}$ for all $n \geq N$. Yet by our old definition, it should be that we can find an index N such that $\left|\frac{1}{n}\right| < \frac{1}{N}$ for all $n \geq N$.

* Alternative Proof $\longrightarrow$

Alternative Proof:

Suppose this new definition is true. Then in particular, there exists N such that for *$\varepsilon = 1$, we have that $|\frac{1}{n}| < 1$ for all $n \geq N$, so $\frac{1}{n}$ converges to 0 (as expected) using our new definition.

Observe then that, by this new definition, it must also have that if $\varepsilon = \frac{1}{2}$ then $|\frac{1}{n}| < \frac{1}{2}$ for $n \geq N$ (i.e., for the same N as before).

However, since $\frac{1}{2} = 1 - \frac{1}{2}$, we have that $|\frac{1}{n}| < 1 - \frac{1}{2}$, or $|\frac{1}{n}| + \frac{1}{2} < 1$, meaning $\frac{1}{n}$ converges to both 0 and $-\frac{1}{2}$ using this new definition

*$\varepsilon = 1$ is arbitrary since we can choose any $\varepsilon > 0$ for this new definition

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Math410: Discussion WS #2

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For this Discussion WS, you'll be exploring a few approaches to proving past homework problems that, while at first seem promising, don't quite get us to the result we want to prove.

1) Recall homework problem #20: Suppose that the function $g : \mathbb{R} \to \mathbb{R}$ is continuous and that $g(x) = 0$ if x is rational. Prove that $g(x) = 0$ for all x in $\mathbb{R}$.

   Proposed setup: Note that $\mathbb{Q}^c$ (i.e., the irrationals) are dense in $\mathbb{R}$. Thus, every number is x is the limit of a sequence in $\mathbb{Q}^c$. In turn, there exists some sequence $a_n \in \mathbb{Q}^c$ that converges to $x_0 \in \mathbb{Q}$.

   Explain why this setup, while true, doesn't help us show $g(x) = 0$ even for x values that are irrational.

2) When working out the scratch work for a proof involving continuity, we often use the strategy of working backwards from where we want to end up. When we use this strategy, it is important that the algebraic steps we take are reversible to use in the actual proof. With this in mind, consider the scratch work below for the following question found in your homework:

Define $f(x) = \sqrt{x}$ for all $x \geq 0$. Verify the $\epsilon - \delta$ criterion for continuity at $x = 4$.

Scratch Work:

$$|\sqrt{x} - 2| < \epsilon$$
$$-\epsilon + 2 < \sqrt{x} < \epsilon + 2$$
$$-\epsilon - 2 < \sqrt{x} < \epsilon + 2$$
$$|\sqrt{x}| < \epsilon + 2$$

–see note[1] below–

$$\sqrt{|x|} < \epsilon + 2$$
$$|x| < (\epsilon + 2)^2$$
$$-(\epsilon + 2)^2 < x < (\epsilon + 2)^2$$
$$-(\epsilon + 2)^2 - 4 < x - 4 < (\epsilon + 2)^2 - 4$$
$$-(\epsilon + 2)^2 - 4 < x - 4 < (\epsilon + 2)^2 + 4$$
$$|x - 4| < (\epsilon + 2)^2 + 4$$

The above scratch work leads us to think we should let $\delta < (\epsilon + 2)^2 + 4$ to bring about $|\sqrt{x} - 2| < \epsilon$, which is what we need to show $f(x) = \sqrt{x}$ is continuous at $x = 4$. Show that the scratch work cannot be easily undone to reach the conclusion we want in our proof.

[1] Note that it can be shown that $|\sqrt{x}| = \sqrt{|x|}$ using the definition of $|x|$ and the fact that the domain of $\sqrt{x}$ is $[0, \infty)$

Math410: Discussion WS #3

Name: UID: Section #:

Read the following text[1] for a brief history on the development of the definition of "continuity."

1) What was something that surprised you about the developments described in the text? Two to three sentences suffices.

[1] https://my.linkpod.site/ContinuityHistory

2) Answer the following (from Exercise 6.1.10):
Let

$$D(x) = \begin{cases} x, & \text{if } x \text{ is rational,} \\ 0, & \text{if } x \text{ is irrational.} \end{cases}$$

a) Use the $\epsilon - \delta$ definition of continuity to prove $D(x)$ is continuous at 0.

b) Suppose $a \neq 0$. Use the $\epsilon - \delta$ definition of continuity to prove $D(x)$ is *not* continuous at a.

Math410: Discussion WS#4

Name: UID: Section #:

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1) Consider the following function:

$$f(x) = \begin{cases} x, & \text{if the point } x \text{ in } [0,1] \text{ is rational} \\ 0, & \text{if the point } x \text{ in } [0,1] \text{ is irrational} \end{cases}$$

a) Prove that $\underline{\int_a^b} f = 0$.

b) Prove that $\overline{\int_a^b} f \geq \frac{1}{2}$.

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Prove that $\int_a^b f \geq \frac{1}{2}$.

SCRAP

$\int_a^b f = \inf\{U(f,P) \mid P \text{ is a partition of } [a,b]\}$

$* a = 0, b = 1$

$$f(x) = \begin{cases} x, & \text{if the point } x \text{ in } [0,1] \text{ is rational} \\ 0, & \text{if the point } x \text{ in } [0,1] \text{ is irrational} \end{cases}$$

Proof: Let P be a regular partition on the interval $[0,1]$. So $[x_i - x_{i-1}] = \frac{b-a}{n} = \frac{1-0}{n} = \frac{1}{n}$ for $1 \leq i \leq n$. Note that on every interval $[x_{i-1}, x_i]$, the smallest possible rational number is x_{i-1}, in which case $f(x_{i-1}) = x_{i-1}$. Otherwise, if x_{i-1} is irrational, then $f(x_{i-1}) = 0$, in which case there exists some other largest rational (by the density of $\mathbb{Q}$) $x_0 \in (x_{i-1}, x_i]$ such that $f(x_0) = x_0$. In either case, $f(x_{i-1}) = x_{i-1}$ serves as the greatest lower bound on each interval, so $m_i \geq x_{i-1}$, and in turn,

$M_i \geq m_i \geq x_{i-1}$.

Thus, by definition of $U(f,P)$, we have the following:

$$U(f,P) = \sum_{i=1}^{n} M_i(x_i - x_{i-1}) \geq \sum_{i=1}^{n} x_{i-1}(x_i - x_{i-1}).$$

But P is a regular partition, so we have that:

$$\sum_{i=1}^{n} x_{i-1}(x_i - x_{i-1}) = \left(\frac{1}{n}\right)\sum_{i=1}^{n}\left(\frac{i-1}{n}\right) = \left(\frac{1}{n^2}\right)\left[\sum_{i=1}^{n} i - \sum_{i=1}^{n} 1\right]$$
$$= \left(\frac{1}{n^2}\right)\left[\frac{n(n+1)}{2} - n\right] = \left(\frac{1}{n^2}\right)\left[\frac{n^2+n-2n}{2}\right] = \frac{n^2-n}{2n^2}.$$

Then for more and more refined partitions of P, or as $n \to \infty$, we have that $U(f,P)$ tends to $\frac{1}{2}$. That is, $\frac{1}{2}$ serves as the greatest lower bound for $U(f,P)$, so $\int_a^b f \geq \frac{1}{2}$. ∎

$* \text{Expository} \rightarrow$

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*Expository: It may be bothersome that the above result followed from choosing P to be a regular partition. That is, does the result hold for other partitions? Yes! The key is noting the step that says "for more and more refined partitions of P" - that is, one can always find a refinement (maybe no longer regular!) such that the above result holds. Hence why in the definition of $\int_a^b f$, we generically create "a" partition.

2) Believe it or not, quite a bit of drama has occurred throughout the development of different ideas in mathematics - let's face it... we all have an ego!

   With this in mind, watch the following[1] video on a historic rivalry of sorts between Sir Isaac Newton and Gottfried Wilhelm Leibniz during the development of calculus.

   What was something that surprised you about the developments described in the video? Two to three sentences suffices.

[1] https://www.youtube.com/watch?v=LN–erHStqA

Math410: Discussion WS #5

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By now, you are probably starting to realize how difficult it is to not only come up with the ideas of a proof, but formally write them down, which often can take just as long, [1] if not longer! We borrow the words of Dr. Thomas C. Hales [2] to define what we mean by a "formal proof":

> A formal proof is a proof in which every logical inference has been checked all the way back to the fundamental axioms of mathematics. All the intermediate logical steps are supplied, without exception. No appeal is made to intuition, even if the translation from intuition to logic is routine. Thus, a formal proof is less intuitive, and yet less susceptible to logical errors.

Still, many believe that for mathematicians, everything related to math not only comes easily, but quickly. With this in mind, watch this [3] video of an interview with Dr. Terence Tao, considered by many to be one of the greatest mathematicians alive.

1) What stood out to you most from Dr. Tao's comments on regarding the idea of struggling with mathematics?

2) What stood out to you most from Dr. Tao's comments regarding proof formalization.

[1] Check out this conference paper if you don't believe me: https://my.linkpod.site/ProofPrograms

[2] https://www.ams.org/notices/200811/200811FullIssue.pdf#page=12.00

[3] https://youtu.be/4bWLBJtrXlI?si=ReRKEiQE_RCvt2Up

Now watch this [4] shortened (to spare you from too much of the monotone audio lol) video Dr. G recorded of himself working through a proof assigned to you for homework. It is important to note a few things as you watch the video: Dr. G overlooked the assumption that $x > 0$ and made a sign error early on in the problem, both of which partially contributed to a lot of overthinking. As a result, after carefully reviewing the theorems, writing out scrap work, engaging in circular thinking, checking logic, writing the formal proof, and reflecting on what was learned from the problem, an hour and a half had passed! Still, the things he overlooked and ideas he played around with led to a deep review of the theorems and logic involved - that is, the time dedicated to this problem was not a waste!

3) In the video, he verbally expresses his ideas, frustrations, and "aha!" moments. What strategies (mathematical or emotional) can you take away from Dr. G's thought process?

4) Some may have looked at this problem and seen the solution in a matter of minutes. Similarly, Dr. Tao shared the following: "Me and my colleagues, we spend a lot of time doing very routine computations, or doing other things that other mathematicians would instantly know how to do, and we don't know how to do them." What can you take away from this observation as you move forward in learning about the art of writing proofs, and more generally, mathematics?

5) Note that there is still a very small adjustment Dr. G should make in his final proof to be more technically accurate. Can you spot it?

   HINT: It has to do with variable notation in his implementation of the Lagrange Remainder Theorem.

[4] `https://my.linkpod.site/WrestlingThroughAProof`

Homework

Math410: Homework

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Instructions

The problems indicated in green are to be turned in via Gradescope and are graded for accuracy. All other problems are suggested. It is recommended to complete **all problems** as this document is where Quiz questions will be pulled from. For instance, if you are asked to complete problem 4c of Section 1.2, you should complete all the problems (unless otherwise indicated) of Section 1.2 to be prepared for the Quiz!

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Math410: Homework

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Intro And Preliminaries

1) *This question is a warm-up from Math310. It is not covered on the quiz or exam in this course but it is to be handed in for homework:* Assume that the product of two integers x and y is even. Show that at least one of the numbers is even.

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Solutions

1) **Proof:** We use a proof by contrapositive. The contrapositive states: If both x and y are odd, then their product xy is odd. If x and y are odd, we can write $x = 2a + 1$ and $y = 2b + 1$, where a, b are integers. Then:

$$xy = (2a + 1)(2b + 1) = 4ab + 2a + 2b + 1 = 2(2ab + a + b) + 1$$

Since $2ab + a + b$ is an integer, xy is odd. By the contrapositive, if xy is even, then at least one of x or y must be even. $\square$

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The Completeness Axiom (1.1)

2) Use induction to prove for any natural number n and any number $r \neq 1$,

$$1 + r + r^2 + \cdots + r^n = \frac{1 - r^{n+1}}{1 - r}.$$

3) For each of the following sets, list three upper bounds and three lower bounds for the set and its supremum/infimum, when such exists:

   a) $\left\{\frac{1}{n} \mid n \in \mathbb{N}\right\}$
   b) $\left\{1 - \frac{1}{3^n} \mid n \in \mathbb{N}\right\}$
   c) $\left\{\cos\left(\frac{n\pi}{3}\right) \mid n \in \mathbb{N}\right\}$ **change to** $\left\{\frac{n}{n+1} : n \in \mathbb{N}\right\}$

   You do not have to prove the supremum/infimum.

4) Suppose that S is a nonempty set of real numbers that is bounded and that $\inf S = \sup S$. Prove that the set S consists of exactly one number.

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Solutions/Hints

2) **Proof:** We prove by induction that for any natural number n and any $r \neq 1$,

$$1 + r + r^2 + \cdots + r^n = \frac{1 - r^{n+1}}{1 - r}$$

Base Case ($n = 1$): The formula holds since both sides equal $1 + r$.
Inductive Step: Assume it holds for $n = k$,

$$1 + r + r^2 + \cdots + r^k = \frac{1 - r^{k+1}}{1 - r}$$

For $n = k + 1$, adding r^{k+1} to both sides gives

$$\begin{aligned} 1 + r + r^2 + \cdots + r^k + r^{k+1} &= \frac{1 - r^{k+1}}{1 - r} + r^{k+1} \\ &= \frac{1 - r^{k+1}}{1 - r} + \frac{r^{k+1}(1 - r)}{1 - r} \\ &= \frac{1 - r^{k+1} + r^{k+1} - r^{k+2}}{1 - r} \\ &= \frac{1 - r^{k+2}}{1 - r} \end{aligned}$$

which is the desired formula for $n = k + 1$. By induction, the result holds for all n. $\square$

3) a) $S = \left\{ \frac{1}{n} \mid n \in \mathbb{N} \right\}$: lower bounds $0, -1, -2$; upper bounds $1, 2, 3$; $\sup S = 1$; $\inf S = 0$.

b) $S = \left\{ 1 - \frac{1}{3^n} \mid n \in \mathbb{N} \right\}$: lower bounds $-1, 0, \frac{1}{2}$; upper bounds $1, 2, 3$; $\sup S = 1$; $\inf S = \frac{2}{3}$.

c) $S = \left\{ \cos\left(\frac{n\pi}{3}\right) \mid n \in \mathbb{N} \right\}$: lower bounds $-1, -2, -3$; upper bounds $1, 2, 3$; $\sup S = 1$; $\inf S = -1$.

4) **Proof:** (By contradiction) Assume that S is a nonempty set of real numbers that is bounded and that $\inf S = \sup S := c$ and assume that there are two distinct numbers a and b in the set S where $a \neq b$.

Without loss of generality, let $a < b$. Since a and b are in S, and $a \leq \sup S = c$ and $b \leq \sup S = c$ and $a \geq \inf S = c$ and $b \geq \inf S = c$. So, $a \geq c$ and $b \leq c$. However, this leads to a contradiction since if $a \neq b$, then either a or b is greater than c or less than c which contradicts the fact that c is both the infimum and supremum of S. $\square$

Math410: Homework

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The Distribution Of The Integers And Rational Numbers (1.2)

5) Using only the Archimedean Property of $\mathbb{R}$, give a direct $\epsilon - N$ proof of the following limit: $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$

6) Using only the Archimedean Property of $\mathbb{R}$, give a direct $\epsilon - N$ proof of the following limit: $\lim_{n\to\infty} \frac{1}{n+5} = 0$

7) Using only the Archimedean Property of $\mathbb{R}$, give a direct $\epsilon - N$ proof of the convergence of $\left\{ \frac{n^2}{n^2+n} \right\}$

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Solutions

5) **Proof:** Let $\epsilon > 0$ be given. Choose an $N \in \mathbb{N}$ such that $N > \frac{1}{\epsilon^2}$ using the Archimedean Property. Then, for all $n \in \mathbb{N}$ if $n \geq N$, then $\frac{1}{n} \leq \frac{1}{N} < \epsilon$, so

$$\left|\frac{1}{\sqrt{n}} - 0\right| = \left|\frac{1}{\sqrt{n}}\right| = \frac{1}{\sqrt{n}} \leq \frac{1}{\sqrt{N}} < \epsilon,$$

as desired. $\square$

6) **Proof:** Let $\epsilon > 0$ be given. Choose an $N \in \mathbb{N}$ such that $N > \frac{1}{\epsilon}$ using the Archimedean Property. Then, for all $n \in \mathbb{N}$ if $n \geq N$, then $\frac{1}{n} \leq \frac{1}{N} < \epsilon$, so

$$\left|\frac{1}{n+5} - 0\right| \leq \left|\frac{1}{n}\right| = \frac{1}{n} \leq \frac{1}{N} < \epsilon,$$

as desired. $\square$

7) **Proof:** The limit of the sequence $\left\{\frac{n^2}{n^2+n}\right\}$ is 1. Let $\epsilon > 0$ be given. Choose an $N \in \mathbb{N}$ such that $N > \frac{1}{\epsilon}$ using the Archimedean Property. Then, for all $n \in \mathbb{N}$ if $n \geq N$, then $\frac{1}{n} \leq \frac{1}{N} < \epsilon$, so

$$\left|\frac{n^2}{n^2+n} - 1\right| = \left|\frac{n^2 - n^2 - n}{n^2+n}\right| = \left|\frac{-n}{n^2+n}\right| = \left|\frac{n}{n^2+n}\right| < \left|\frac{n}{n^2}\right| = \frac{1}{n} < \frac{1}{N} < \epsilon,$$

as desired. $\square$

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Inequalities And Identities (1.3)

8) Prove that $|a + b + c| \leq |a| + |b| + |c|$ for all $a, b, c \in \mathbb{R}$. Hint: do not look at eight cases, but rather apply the triangle inequality twice.
Then use induction to prove $|a_1 + \cdots + a_n| \leq |a_1| + \cdots + |a_n|$ for all $n \in N$ and $a_i \in \mathbb{R}$.

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Solutions

8) **Proof:** Let $a, b, c \in \mathbb{R}$. Then

$$\begin{aligned} |a+b+c| &= |(a+b)+c| && \text{by associative property of } \mathbb{R} \\ &\leq |a+b|+|c| && \text{by the triangle inequality for two elements of } \mathbb{R} \\ &\leq |a|+|b|+|c| && \text{by the triangle inequality for two elements of } \mathbb{R} \end{aligned}$$

$\square$

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The Convergence Of Sequences (2.1)

9) Consider three sequences $\{a_n\}, \{b_n\}$, and $\{s_n\}$ such that $a_n \leq s_n \leq b_n$ for all n and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = s$. Prove that $\lim_{n\to\infty} s_n = s$.

10) Prove that the sequence $\{c_n\}$ converges to c if and only if the sequence $\{c_n - c\}$ converges to 0.

11) Let $\{a_n\}$ be a sequence of real numbers. Suppose that for each $M > 0$ there is an index N such that $a_n > M$ for all indices $n \geq N$. When this is so, the sequence $\{a_n\}$ is said to **diverge to infinity**, and we write $\lim_{n\to\infty} a_n = \infty$. Prove that $\lim_{n\to\infty} \left[n^3 - 4n^2 - 100n\right] = \infty$.

12) For a sequence $\{a_n\}$ of positive numbers show $\lim_{n\to\infty} a_n = \infty$ if and only if $\lim_{n\to\infty} \left[\frac{1}{a_n}\right] = 0$.

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Solutions/Hints

9) **Proof:** Let $a_n \leq s_n \leq b_n$ and $a_n \to s$ and $b_n \to s$. By the definition of convergence, there is a $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$, $|a_n - s| < \epsilon$. Or equivalently,

$$s - \epsilon < a_n < s + \epsilon$$

$n \geq N_1$. Similarly, there is $N_2 \in \mathbb{N}$ such that for all $n \geq N_2$, $|b_n - s| < \epsilon$. Or equivalently,

$$s - \epsilon < b_n < s + \epsilon$$

$n \geq N_2$. Take $N = \max\{N_1, N_2\}$ and then for all $n \geq N$, combining the assumption that $a_n \leq s_n \leq b_n$ for all n with the two above yields

$$s - \epsilon < a_n \leq s_n \leq b_n < s + \epsilon$$

for all $n \geq N$, and thus $s - \epsilon < s_n < s + \epsilon$, or that

$$|s_n - s| < \epsilon$$

for all $n \geq N$, as needed for convergence.

$\square$

10) This follows directly from the fact that $|c_n - c| = |(c_n - c) - 0|$

11) First note that the polynomial sequence can be rewritten so that $n^3 - 4n^2 - 100n = n(n - (2 + 2\sqrt{26})(n - (2 - 2\sqrt{26})$. Let's develop some scrap work to show that this polynomial diverges to infinity, that is, grows without bound. By the Archimedian Property, assume $n - (2 + 2\sqrt{26}) > 1$ (note that choosing 1 is arbitrary). Performing some algebraic steps to solve for n, we find that this means $n > 13$, and $n - (2 - 2\sqrt{26}) > 21$. Thus, we have the following:

$$(13)(1)(21) < |(n)(n - (2 + 2\sqrt{26})(n - (2 - 2\sqrt{26})|$$

However, this only will help us show that the polynomial is unbounded "up to" $M < (13)(1)(21) = 273$. That is, we can find values of n the make the polynomial as big as we want, all the way up 273. To cover when $M \geq 273$, let's still assume $n > 13$ (from above), but *also* that $n > M$, giving us the following inequality:

$$(M)(1)(21) < |(n)(n - (2 + 2\sqrt{26})(n - (2 - 2\sqrt{26})|$$

In turn, we have the following:

$$M < (M)(1)(21) < |(n)(n - (2 + 2\sqrt{26})(n - (2 - 2\sqrt{26})|$$

To ensure we cover both when $M < 273$ and when $M \geq 273$, we'll assume $n > max(13, M)$, satisfying both conditions used above. In turn, we arrive at the fact that for each $M > 0$ (we covered *all* choices of M), there is an index N (namely $N > max(13, M)$)[1], such that $a_n > M$ for all indices $n \geq N$. **Now write the proof using this scrap work!**

[1]Note we did *not* write $N = max(13, M)$ as M may not be an integer.

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12) For a sequence $\{a_n\}$ of positive numbers show that $\lim_{n\to\infty} a_n = \infty$ if and only if $\lim_{n\to\infty}\left[\frac{1}{a_n}\right] = 0$

Proof: First, assume that $\lim_{n\to\infty} a_n = \infty$. By definition, this means for all $M > 0$, there is an $N \in \mathbb{N}$ such that for all $n \geq N$, we have $a_n > M$, or that $\frac{1}{a_n} < \frac{1}{M}$. Let $\epsilon > 0$ be given and we can define $M := \frac{1}{\epsilon}$. Thus, from the N found from $\{a_n\}$ diverging to ∞, there is an $N \in \mathbb{N}$ such that for all $n \geq N$ then

$$\left|\frac{1}{a_n} - 0\right| = \frac{1}{a_n} < \frac{1}{M} < \epsilon$$

as needed. Thus, $\lim_{n\to\infty}\left[\frac{1}{a_n}\right] = 0$.

For the other direction, the proof is similar except with $\epsilon := \frac{1}{M}$. $\square$

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Sequences And Sets (2.2)/The Monotone Convergence Theorem (2.3)

13) For each of the following statements, determine whether it is true or false and justify your answer.

   a) Every bounded sequence converges.
   b) A convergent sequence of positive numbers has a positive limit.
   c) The sequence $\{n^2+1\}$ converges.
   d) A convergent sequence of rational numbers has a rational limit.
   e) The limit of a convergent sequence in the interval (a,b) also belongs to (a,b).

14) Suppose that the sequence $\{a_n\}$ is monotone. Prove that $\{a_n\}$ converges if and only if $\{a_n^2\}$ converges. Show that this result does not hold without the monotonicity assumption.

15) Let $\{b_n\}$ be a bounded sequence of nonnegative numbers and r be any number such that $0 \le r < 1$. Define

$$s_n = b_1 r + b_2 r^2 + \cdots + b_n r^n \quad \text{for every index } n$$

Use the Monotone Convergence Theorem to prove that the series $\{s_n\}$ converges.

16) Submit an outline of all the main steps within the proof and main properties used: Let c be a number such that $|c| < 1$. Then

$$\lim_{n\to\infty} c^n = 0$$

Proof: The case $c = 0$ is clear since we are then considering the constant sequence all of whose terms are 0 . Moreover, since $|c^n| = |(-c)^n|$, the case $c < 0$ follows from the case $c > 0$. So assume $c > 0$. Since $0 < c < 1$, $\{c^n\}$ is a monotonically decreasing sequence bounded below by 0 . According to the Monotone Convergence Theorem, the sequence $\{c^n\}$ converges to a number ℓ where

$$\ell = \inf\{c^n \mid n \text{ in } \mathbb{N}\}$$

We must have $\ell = 0$, since otherwise, because $c > 0$, we have

$$c^n = \frac{c^{n+1}}{c} \ge \frac{\ell}{c} \quad \text{for every index } n$$

so that ℓ/c is a lower bound for the sequence. Thus, ℓ/c is a lower bound for the sequence, and it is larger than ℓ since $0 < c < 1$. Hence ℓ is not the greatest lower bound for the sequence. This contradiction shows that $\ell = 0$. $\square$

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Solutions/Hints

13) All statements are false. Find counterexamples!
14) What does convergence also imply about a sequence?
15) Consider using the Triangle Inequality!
16) N/A

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The Sequential Compactness Theorem (2.4)

17) Prove that a sequence $\{a_n\}$ does not converge to the number a if and only if there is some $\epsilon > 0$ and a subsequence $\{a_{n_k}\}$ such that

$$|a_{n_k} - a| \geq \epsilon \quad \text{for every index } k.$$

18) For each of the following statements, determine whether it is true or false and justify your answer.

a) A subsequence of a bounded sequence is bounded.

b) A subsequence of a monotone sequence is monotone.

c) A subsequence of a convergent sequence is convergent.

d) A sequence converges if it has a convergent subsequence.

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Solutions/Hints

17) Define a specific subsequence where this is true! To prove the converse direction, consider writing a proof by contradiction.
18) Only one of these statements is false.

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Continuity (3.1)

19) Define

$$f(x) = \begin{cases} 11 & \text{if } 0 \le x \le 1 \\ x & \text{if } 1 < x \le 2. \end{cases}$$

At what points is the function $f : [0, 2] \to \mathbb{R}$ continuous? Justify your answer with a proof.

20) Suppose that the function $g : \mathbb{R} \to \mathbb{R}$ is continuous and that $g(x) = 0$ if x is rational. Prove that $g(x) = 0$ for all x in $\mathbb{R}$

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Solutions/Hints

19) You may rely on the fact that all polynomials are continuous.

20) Recall that for *every type* of number x, there exists a sequence of rational numbers that converge to x.

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The Extreme Value Theorem (3.2)

21) Is it true that if $f : [a, b] \to \mathbb{R}$ has a maximum and minimum value, then f must be continuous? Justify your answer.

Solutions/Hints

21) False. Find a counterexample!

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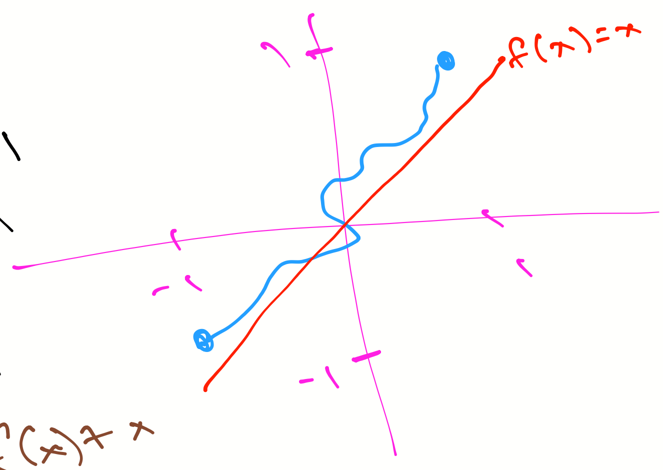
The Intermediate Value Theorem (3.3)

22) For a function $f : D \to \mathbb{R}$, a solution of the equation

$$f(x) = x, \quad x \text{ in } D$$

is called a fixed point of f. A fixed point corresponds to a point at which the graph of the function f intersects the line $y = x$. If $f : [-1, 1] \to \mathbb{R}$ is continuous, $f(-1) > -1$, and $f(1) < 1$, prove that $f : [-1, 1] \to \mathbb{R}$ has a fixed point.



SCRAP

$f(-1) > -1 \iff 0 < -f(-1) - 1$

$f(1) < 1 \iff 0 < -f(1) + 1$

$-f(1) + 1 < 0 < -f(-1) - 1$

let $g(x) = -f(x) + x$
$= x - f(x)$

Proof: Let's define $g : [-1,1] \to \mathbb{R}$ as $g(x) = x - f(x)$. Since f is continuous, this implies g is also continuous since it is the difference of two continuous functions, with $g(-1) = -1 - f(-1) < -1 + 1 = 0$. Likewise, $g(1) > 0$. Then by the IVT there exists $x_0 \in (-1, 1)$ such that $g(x_0) = 0$. That is, $g(x_0) = x_0 - f(x_0) = 0$, or $x_0 = f(x_0)$. That is, x_0 is a fixed point of f. ∎

↳ at least one!

Solutions/Hints

22) $f(-1)+1>0$ and $f(1)-1<0$.

Uniform Continuity (3.4), The $\epsilon - \delta$ Criterion For Continuity (3.5)

Note: Unless otherwise indicated, you may use the sequences or the $\epsilon - \delta$ definitions of continuity and uniform continuity in the following problems.

23) Prove that if $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$ are uniformly continuous and α is any number, then the sum $\alpha f + g : D \to \mathbb{R}$ is also uniformly continuous (using the sequences definition).

24) Define $f : \mathbb{R} \to \mathbb{R}$ as $f(x) = x^3$ for all x.

   a) Prove that the function $f : \mathbb{R} \to \mathbb{R}$ is not uniformly continuous (using the sequences definition).

   b) Verify the $\epsilon - \delta$ criterion for continuity at each point x_0.

25) A function $f : D \to \mathbb{R}$ is called a Lipschitz function if there is some nonnegative number C such that

$$|f(u) - f(v)| \leq C|u - v| \quad \text{for all points } u \text{ and } v \text{ in } D.$$

   a) Prove that if $f : D \to \mathbb{R}$ is a Lipschitz function, then it is uniformly continuous (using the sequences definition).

   b) Prove that a Lipschitz function satisfies the $\epsilon - \delta$ criterion on D (using the $\epsilon - \delta$ definition).

26) a) Provide an example illustrating that it is not necessarily the case that if $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$ are each uniformly continuous, then so is the product $fg : D \to \mathbb{R}$.

   b) Suppose that the functions $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$ are uniformly continuous and bounded. Prove that the product $fg : D \to \mathbb{R}$ also is uniformly continuous.

27) Define $f(x) = \sqrt{x}$ for all $x \geq 0$. Verify the $\epsilon - \delta$ criterion for continuity at $x = 4$. $\to$ proof on next page!

(26) b) SCRAP

Want to show: $|a-b| < \delta \to |f(a)g(a) - f(b)g(b)| < \varepsilon$

Know:
- $|a-b| < \delta_1 \to |f(a) - f(b)| < \varepsilon$
- $|a-b| < \delta_2 \to |g(a) - g(b)| < \varepsilon$
- $|g(x)| \leq W$
- $|f(x)| \leq M$

$S = \max(W, M)$

Observe: $|g(a)||f(a) - f(b)| + |f(b)||g(a) - g(b)| < \cancel{W}^{\,S}\left(\frac{\varepsilon}{2S}\right) + \left(\frac{\varepsilon}{2S}\right)\cancel{M}^{\,S} < \varepsilon$

$|f(a)g(a) - f(b)g(b)| \leq |g(a)||f(a) - f(b)| + |f(b)||g(a) - g(b)|$ (Δ ineq.)

27) SCRAP

$|x-4| < \delta \to |\sqrt{x} - 2| < \varepsilon$

$f(x)$ $\quad f(4)$

* Note that $|ab| = |a||b|$

$|\sqrt{x}-2||\sqrt{x}+2| < \delta$

$$|\sqrt{x}-2| < \frac{\delta}{|\sqrt{x}+2|} \leq \frac{\delta}{2} \overset{\text{want}}{<} \varepsilon$$

* we want $\delta = 2\varepsilon$

Proof: Let $\varepsilon > 0$. Then we wish to show that for some $\delta > 0$, if $|x-4| < \delta$, then $|\sqrt{x}-2| < \varepsilon$. Let $\delta = 2\varepsilon$. Then we have that $|x-4| = |\sqrt{x}-2||\sqrt{x}+2| < 2\varepsilon$, so $|\sqrt{x}-2| < \varepsilon$. ∎

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Solutions/Hints

23) First write out *exactly* what it is you want to show.

24) a) Experiment with pairs of sequences!

 b) Make an assumption about δ along the way, and use the $\delta = min(?, ?)$ strategy.

25) a) Think about using the Comparison Lemma.

 b) What does it mean for $\lim_{n\to\infty}[u_n - v_n] = 0$ in terms of the ϵ and N?

26) a) Again, experiment with pairs of sequences!

 b) **Proof:** Assume that f and g are uniformly continuous. Let $\{u_n\}$ and $\{v_n\}$ be sequences in D with $\lim\limits_{n\to\infty}[u_n - v_n] = 0$. Since f and g are bounded we have that there exist numbers M_1 and M_2 with $|f(x)| \leq M_1$ and $|g(x)| \leq M_2$ for all $x \in D$. Define $M = \max\{M_1, M_2\}$ (note that you can complete this proof without defining a common max). Then for each index n,

$$\begin{aligned}
|f(u_n)g(u_n) - f(v_n)g(v_n)| &= |f(u_n)g(u_n) - f(u_n)g(v_n) + f(u_n)g(v_n) - f(v_n)g(v_n)| \\
&= |f(u_n)(g(u_n) - g(v_n)) + g(v_n)(f(u_n) - f(v_n))| \\
&\leq |f(u_n)[g(u_n) - g(v_n)]| + |g(v_n)[f(u_n) - f(v_n)]| \\
&\leq M_1 |g(u_n) - g(v_n)| + M_2 |f(u_n) - f(v_n)| \\
&\leq M(|g(u_n) - g(v_n)| + |f(u_n) - f(v_n)|),
\end{aligned}$$

 Since f and g are uniformly continuous we have

$$\lim_{n\to\infty}[g(u_n) - g(v_n)] = \lim_{n\to\infty}[f(u_n) - f(v_n)] = 0$$

 so that by the Comparison Lemma, $\lim\limits_{n\to\infty}[f(u_n)g(u_n) - f(v_n)g(v_n)] = 0$ as well. Thus fg is uniformly continuous. $\square$

27) Note that $|x-4| < \delta$ implies $|\sqrt{x}-2||\sqrt{x}+2| < \delta$. Now, find some expression that doesn't involve x, but is still in terms of δ, such that $\frac{\delta}{|\sqrt{x}+2|} <$ said expression.

Images Of Inverses And Monotone Functions (3.6)

28) A function $f : \mathbb{R} \to \mathbb{R}$ is said to be odd provided that $f(-x) = -f(x) \quad$ for all x. Prove that if $f : \mathbb{R} \to \mathbb{R}$ is odd, and the restriction of this function to the interval $[0, \infty)$ is strictly increasing, then $f : \mathbb{R} \to \mathbb{R}$ itself is strictly increasing.

29) A good study habit for understanding and writing proofs is to create an outline of the main steps and properties used in a proof presented in the textbook. Here is an example of a proof outline for the Sum Property for sequences:

    - Sequences $\{a_n\}$ and $\{b_n\}$ converge $\to$ use the definition to control the difference $|a_n - a|$ and $|b_n - b|$ each by $\epsilon/2$.
    - Choose N that is bigger than the 2 N's from a_n and b_n
    - Set up main difference $|a_n + b_n - (a + b)|$ and use triangle inequality

    Read the following proposition and its associated proof, which was used in chapter 3.6. Then create an outline of all the main steps within the proof and main properties used.

    Note: The following proof is meant to be in section 3.6 to help build the following theorem: Let I be an interval and suppose that the function $f : I \to \mathbb{R}$ is strictly monotone. Then the inverse function $f^{-1} : f(I) \to \mathbb{R}$ is continuous. That is, we cannot just assume continuity of $f(x) = x^{1/3}$ just yet!

    Proposition: The inverse of $f(x) = x^3$ is continuous.

    Proof. Note f is a polynomial and thus continuous. And note that f is strictly increasing by properties of inequalities, for $u < v$ then $u^3 < v^3$. So we can use the theorem that any strictly monotone $f : D \to \mathbb{R}$ is one-to-one and thus has an inverse. Fix $x_0 \in \mathbb{R}$. Let $\{x_n\} \in \mathbb{R}$ with $x_n \to x_0$. Let $y_n = f^{-1}(x_n)$ and $y_0 = f^{-1}(x_0)$. So,

    $$x_n = f(y_n) = y_n^3 \text{ and } x_0 = f(y_0) = y_0^3$$

    Additionally, since $x_n \to x_0$, then $y_n^3 \to y_0^3$. Let $\epsilon > 0$ be given. Choose $\delta = \min\left((y_0 + \epsilon)^3 - y_0^3, y_0^3 - (y_0 - \epsilon)^3\right)$. Note $\delta > 0$ since $\epsilon > 0$ yielding $(y_0 + \epsilon)^3 - y_0^3 > 0$ and $y_0^3 - (y_0 - \epsilon)^3 > 0$. Furthermore, since $y_n^3 \to y_0^3$, there exists N such that for all $n \geq N, |y_n^3 - y_0^3| < \delta$. Rewriting this inequality,

    $$\begin{aligned} y_0^3 - \delta < &y_n^3 < \delta + y_0^3 \\ (y_0 - \epsilon)^3 < &y_n^3 < (y_0 + \epsilon)^3 \\ y_0 - \epsilon < &y_n < y_0 + \epsilon \end{aligned}$$

    So, $f^{-1}(x)$ is continuous.

    $\square$

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Solutions/Hints

28) Observe that $-f(a) < -f(b)$. Now, how can you rewrite this inequality knowing f is odd?

29) I'll leave this to you!

The Algebra Of Derivatives (4.1), Differentiating Inverses And Compositions (4.2), And The Mean Value Theorem (4.3)

30) Use the definition of derivative to compute the derivative of the following functions at $x = 1$:

    a) $f(x) = \sqrt{x+1}$ for all $x > 0$.

    b) $f(x) = 1/\left(1+x^2\right)$ for all x.

31) Let I and J be open intervals, and the functions $f : I \to \mathbb{R}$ and $h : J \to \mathbb{R}$ have the property that $h(J) \subseteq I$, so the composition $f \circ h : J \to \mathbb{R}$ is defined. Prove that if x_0 is in J, and $h : J \to \mathbb{R}$ is continuous at x_0 with $h(x) \neq h(x_0)$ if $x \neq x_0$, and $f : I \to \mathbb{R}$ is differentiable at $h(x_0)$, then

$$\lim_{x \to x_0} \frac{f(h(x)) - f(h(x_0))}{h(x) - h(x_0)} = f'(h(x_0)).$$

see next page for proof outline!

32) Suppose that the function $f : (0, \infty) \to \mathbb{R}$ is differentiable and let $c > 0$. Now define $g : (0, \infty) \to \mathbb{R}$ by $g(x) = f(cx)$ for $x > 0$. Just using the definition of derivative, prove that $g'(x) = cf'(cx)$ for $x > 0$.

33) Prove that the following equation has exactly one solution:

$$x^5 + 5x + 1 = 0, \quad -1 < x < 0$$

34) Let $g : \mathbb{R} \to \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ be differentiable functions and suppose that

$$g(x)f'(x) = f(x)g'(x) \quad \text{for all } x.$$

If $g(x) \neq 0$ for all x in $\mathbb{R}$, show that there is some c in $\mathbb{R}$ such that $f(x) = cg(x)$ for all x in $\mathbb{R}$.

35) Suppose that $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are each differentiable and that

$$\begin{cases} f'(x) = g(x) \quad \text{and} \quad g'(x) = -f(x) \quad \text{for all } x \\ f(0) = 0 \quad \text{and } g(0) = 1. \end{cases}$$

Prove that

$$[f(x)]^2 + [g(x)]^2 = 1 \quad \text{for all } x.$$

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3i)

$\lim_{x \to x_0} h(x) = h(x_0)$.

Let $y = h(x)$ and $y_0 = h(x_0)$.

Then $f'(h(x_0)) = f'(y_0) = \lim_{y \to y_0} \frac{f(y) - f(y_0)}{y - y_0}$

$= z_0$.

Then by comp. of limits, $f'(h(x_0)) = \lim_{x \to x_0} \frac{f(h(x)) - f(h(x_0))}{h(x) - h(x_0)}$

3j) $\lim_{x \to x_0} cx = cx_0$ (b/c cx is a poly.)

Let $y = h(x) = cx$ and $y_0 = h(x_0) = cx_0$.

Then $g'(x_0) = [f(h(x_0))]' = [f(y_0)]' = \lim_{y \to y_0} \frac{f(y) - f(y_0)}{y - y_0}$.

Then by comp. prop. of limits, $[f(h(x_0))]' = \lim_{x \to x_0} \frac{f(h(x)) - f(h(x_0))}{h(x) - h(x_0)}$

$= \lim_{x \to x_0} \frac{f(cx) - f(cx_0)}{cx - cx_0}$

$\to c[f(cx_0)]' = \lim_{x \to x_0} \frac{f(cx) - f(cx_0)}{x - x_0}$

$= \lim_{x \to x_0} \frac{g(x) - g(x_0)}{x - x_0} = g'(x_0)$.

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36) Let D be the set of nonzero real numbers. Suppose that the functions $g : D \to \mathbb{R}$ and $h : D \to \mathbb{R}$ are differentiable and that

$$g'(x) = h'(x) \quad \text{for all } x \text{ in } D.$$

Do the functions $g : D \to \mathbb{R}$ and $h : D \to \mathbb{R}$ differ by a constant? Explain your reasoning with a proof or justification for a counterexample. *Hint: Is D an interval?*

Solutions/Hints

30) a) Consider using the conjugate.
    b) *Carefully* work through the algebra.
31) We can say that as $x \to x_0$, $h(x) \to h(x_0)$. Why? In turn, we can make a substitution.
32) Use the composition property of limits (see Theorem 3.37 in Section 3.6, and example 3.38).
33) Use the Intermediate Value Theorem, followed by Rolle's Theorem.
34) What does the expression $g(x)f'(x) - f(x)g'(x)$ remind you of?
35) Consider using the Chain Rule.
36) No. Find an counterexample!

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The Cauchy Mean Value Theorem (4.4), Darboux Sums: Upper And Lower Integrals (6.1)

37) Suppose $a < b$ are positive real numbers and $f : [a, b] \to \mathbb{R}$ is continuous and its restriction to (a, b) is differentiable. Use the Cauchy Mean Value Theorem to prove that there is a real number $c \in (a, b)$ for which

$$\frac{af(b) - bf(a)}{a - b} = f(c) - cf'(c).$$

38) Consider the partition $P = \{0, \frac{1}{4}, \frac{1}{2}, 1\}$ of the interval $[0, 1]$. Compute $L(f, P)$ and $U(f, P)$ for $f(x) = x$ for all $x \in [0, 1]$.

39) Suppose that the two bounded functions $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ have the property that

$$g(x) \leq f(x) \quad \text{for all } x \text{ in } [a, b].$$

a) For P a partition of $[a, b]$, show that $L(g, P) \leq L(f, P)$.

b) Use part (a) to show that $\underline{\int_a^b} g \leq \underline{\int_a^b} f$.

37) Let $g(x) = \frac{f(x)}{x}$ and $h(x) = \frac{1}{x}$.

We will apply the CMVT, namely, since a, b are positive, then the interval $[a,b]$ does not contain 0. Note that both g and h are continuous on $[a,b]$. Moreover, they are differentiable on (a,b).

Now note that $g'(x) = \frac{x f'(x) - f(x)}{x^2}$ and $h'(x) = \frac{-1}{x^2}$, again with $x \neq 0$.

By the CMVT, there is a $c \in (a,b)$ for which:

$$\frac{g(b) - g(a)}{h(b) - h(a)} = \frac{g'(c)}{h'(c)}.$$

Using the above definitions of g and h, and simplifying, we have:

$$\frac{\frac{f(b)}{b} - \frac{f(a)}{a}}{\frac{1}{b} - \frac{1}{a}} = \frac{c f'(c) - f(c)}{-\frac{1}{c^2}}.$$

And in turn, $\frac{a f(b) - b f(a)}{a - b} = f(c) - c f'(c)$. ∎

Solutions/Hints

37) This one is quite challenging! Try to come up with two functions g and h such that $g(a) = bf(x)$, $g(b) = af(x)$, $h(a) = b$, and $h(b) = a$ (note that h is not be in terms of f). *Then* use the Cauchy Mean Value Theorem.

38) *Carefully* apply the definitions of $L(f, P)$ and $U(f, P)$, while also using what you know about $f(x) = x$.

39) a) **Proof:** Recall that the lower sum $L(f, P)$ is given by:

$$L(f,P) \equiv \sum_{i=1}^{n} M_i(f,P)\,(x_i - x_{i-1})$$

Now, since $g(x) \leq f(x)$ for all x in $[a, b]$, it follows that $M_i(g, P) \leq M_i(f, P)$ for each subinterval $[x_{i-1}, x_i]$ by the definition of infimum. Therefore, we can write:

$$L(g,P) = \sum_{i=1}^{n} M_i(g,P)\,(x_i - x_{i-1}) \leq \sum_{i=1}^{n} M_i(f,P)\,(x_i - x_{i-1}) = L(f,P)$$

This shows that $L(g, P) \leq L(f, P)$ for any partition P of the interval $[a, b]$. ■

b) **Proof:** By definition of a lower integral,

$$\underline{\int_a^b} f \equiv \sup\{L(f,P) \mid P \text{ a partition of the interval } [a,b]\}$$

and

$$\underline{\int_a^b} g \equiv \sup\{L(g,P) \mid P \text{ a partition of the interval } [a,b]\}$$

Since from part a we have $L(g, P) \leq L(f, P)$, it follows directly from definition that

$$\underline{\int_a^b} g \leq \underline{\int_a^b} f.$$

■

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The Archimedes-Riemann Theorem (6.2), Additivity, Monotonicity, And Linearity (6.3), Continuity And Integrability (6.4)

40) For each of the following statements, determine whether it is true or false and justify your answer.

   a) If $f : [a, b] \to \mathbb{R}$ is integrable and $\int_a^b f = 0$, then $f(x) = 0$ for all x in $[a, b]$.

   b) If $f : [a, b] \to \mathbb{R}$ is integrable, then $f : [a, b] \to \mathbb{R}$ is continuous.

   c) If $f : [a, b] \to \mathbb{R}$ is integrable and $f(x) \geq 0$ for all x in $[a, b]$, then $\int_a^b f \geq 0$.

   d) A continuous function $f : (a, b) \to \mathbb{R}$ defined on an open interval (a, b) is bounded.

   e) A continuous function $f : [a, b] \to \mathbb{R}$ defined on a closed interval $[a, b]$ is bounded.

41) Recall that $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$. Use the Archimedes-Riemann Theorem to show $f(x) = x^2$ is integrable on $[a, b]$ for $a, b > 0$ and that $\int_a^b x^2 dx = \frac{b^3 - a^3}{3}$.

42) Suppose that the functions $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ are integrable. Show that there is a sequence $\{P_n\}$ of partitions of $[a, b]$ that is an Archimedean sequence of partitions for f on $[a, b]$ and also an Archimedean sequence of partitions for g on $[a, b]$.

43) **Definition:** For a partition $P = \{x_0, \ldots, x_n\}$ on $[a, b]$, we define the gap of P to be the length of the largest partition interval of P; that is,

$$\text{gap } P \equiv \max_{i \leq i \leq n} [x_i - x_{i-1}]$$

   a) For a partition $P = \{x_0, \ldots, x_n\}$ of the interval $[a, b]$, show that

$$\sum_{i=1}^{n} [x_i - x_{i-1}]^2 \leq [b - a] \cdot \text{ gap } P.$$

   b) Suppose that the function $f : [a, b] \to \mathbb{R}$ is Lipschitz; that is, that there is a constant $c \geq 0$ such that $|f(u) - f(v)| \leq c|u - v|$ for all points u, v in $[a, b]$. For a partition P of $[a, b]$, prove that

$$0 \leq U(f, P) - L(f, P) \leq c[b - a] \cdot \text{gap } P.$$

   Hint: Use the EVT and part a.

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Solutions/Hints

40) a) False. Find a counterexample!
 b) False. A step function is integrable
 c) True. Use the definitions of the lower and upper sums.
 d) False. Find a counterexample!
 e) True by the EVT!

41) Use a regular partition, and the fact that x^2 is monotonically increasing. This problem will make you appreciate our integration formulas lol!

42) Define the union of two partitions. We claim that this union is a refinement of each of these partitions simultaneously. Why?

43) a) By the definitions of a partition and its gap, we have $0 < x_i - x_{i-1} \leq \operatorname{gap} P$ for all i. Use this to show how $(x_i - x_{i-1})^2$ compares to $(x_i - x_{i-1}) \cdot \operatorname{gap} P$, followed by taking the summation of the relationship you find from $i = 1$ to n.

 b) **Proof:** From a previous homework, we showed that Lipschitz functions are continuous. Therefore, by EVT, the function f attains both a maximum and a minimum on each subinterval $[\ x_{i-1}, x_i]$. Let $u_i, v_i \in [x_{i-1}, x_i]$ be points where these maximum and minimum values occur, respectively. Then,

$$M_i - m_i = f(v_i) - f(u_i) \leq c\,|v_i - u_i| \leq c\,(x_i - x_{i-1})$$

for each i, where the Lipshitz property was used for $f(v_i) - f(u_i) \leq c\,|v_i - u_i|$. Using this in the formula below, we get

$$U(f,P) - L(f,P) = \sum_{i=1}^{n} (M_i - m_i)(x_i - x_{i-1}) \leq c \sum_{i=1}^{n} (x_i - x_{i-1})^2 \leq c(b-a) \cdot \operatorname{gap} P$$

as needed. $\square$

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The First Fundamental Theorem Of Calculus: Integrating Derivatives (6.5), The Second Fundamental Theorem Of Calculus: Differentiating Integrals (6.6)

44) Calculate the following derivatives:

a) $\frac{d}{dx}\left(\int_1^{e^x} \ln t dt\right)$

b) $\frac{d}{dx}\left(\int_{-x}^{x} e^{t^2} dt\right)$

45) Suppose that the function $f : \mathbb{R} \to \mathbb{R}$ is differentiable. Define the function $H : \mathbb{R} \to \mathbb{R}$ by

$$H(x) = \int_{-x}^{x} [f(t) + f(-t)]dt \quad \text{for all } x.$$

Find $H''(x)$.

46) Suppose that the function $f : \mathbb{R} \to \mathbb{R}$ has a continuous second derivative. Prove that

$$f(x) = f(0) + f'(0)x + \int_0^x (x-t)f''(t)dt \quad \text{for all } x.$$

47) Let the function $f : [a, b] \to \mathbb{R}$ be continuous. Suppose that the function $F : [a, b] \to \mathbb{R}$ is continuous, that $F : (a, b) \to \mathbb{R}$ is differentiable, and that $F'(x) = f(x)$ for all x in (a, b).

a) Use the Second Fundamental Theorem of Calculus (FTC) to prove that

$$\frac{d}{dx}\left[F(x) - \int_a^x f\right] = 0 \quad \text{for all } x \text{ in } (a, b)$$

b) Use part a to derive a new proof of the first FTC (Integrating Derivatives).

Solutions/Hints

44) a) Recall that the antiderivative of $ln(t)$ is $tln(t) - t$.

b) There is no easy antiderivative of e^{t^2}. One way to get around this is by pretending you know the antiderivative, call this $F(x)$, and carefully go through the integration process (i.e., applying the Fundamental Theorem of Calculus).

45) Use a similar strategy as to that of 44b, and make use of the Chain Rule!

46) Try using integration by parts.

47) a) Use a similar strategy as to that of 44b, yet again.

b) **Proof:** Since f is continuous, the FTC2 implies that $\dfrac{d}{dx}\displaystyle\int_a^x f = f(x)$ for all x in (a, b), thus

$$\frac{d}{dx}\left[F(x) - \int_a^x f\right] = F'(x) - f(x) = f(x) - f(x) = 0$$

as needed, since we are given that $F'(x) = f(x)$.
Applying the the Identity Criterion, there exists a constant c such that

$$F(x) - \int_a^x f = c$$

for all x in (a, b). To get continuity at the endpoints note that f is assumed to be continuous, so f is integrable and thus $\displaystyle\int_a^x f$ is continuous on $[a, b]$. So the difference $F(x) - \displaystyle\int_a^x f$ is continuous on $[a, b]$ since it is the difference of two continuous functions; therefore

$$F(x) - \int_a^x f = c$$

for all x in $[a, b]$ (endpoints included). Plugging in $x = a$ shows that $c = F(a)$, and then taking $x = b$ gives

$$F(b) - \int_a^b f = F(a)$$

or that $\displaystyle\int_a^b f = F(b) - F(a)$ for all x in $[a, b]$, proving FTC1.

∎

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Taylor Polynomials (8.1), The Lagrange Remainder Theorem (8.2), The Convergence Of Taylor Polynomials (8.3), The Cauchy Integral Remainder Theorem (8.5)

48) For each of the following pairs of functions, determine its highest order of contact at the indicated point:

a) $f(x) = x^2$ and $g(x) = \sin x$ for all $x; x_0 = 0$.

b) $f(x) = e^{x^2}$ and $g(x) = 1 + 2x^2$ for all $x; x_0 = 0$.

49) Define $f(x) = x^6 e^x$ for all x. Find the sixth Taylor polynomial for the function f at $x = 0$.

50) Prove that

$$1 + \frac{x}{3} - \frac{x^2}{9} < (1 + x)^{\frac{1}{3}} < 1 + \frac{x}{3} \quad \text{if } x > 0$$

51) A number x_0 is said to be a *root of order* k of the polynomial p provided that k is a natural number with

$$p(x) = (x - x_0)^k r(x),$$

where r is a polynomial, and $r(x_0) \neq 0$.

Prove that if x_0 is a root of order k of the polynomial p, then

$$p(x_0) = p'(x_0) = \cdots = p^{(k-1)}(x_0) = 0 \quad \text{and} \quad p^{(k)}(x_0) \neq 0.$$

Note that the converse can also be proven!

52) Suppose that the function $F : \mathbb{R} \to \mathbb{R}$ has derivatives of all orders, and that

$$\begin{cases} F''(x) - F'(x) - F(x) = 0, & \text{for all } x, \\ F(0) = 1, \\ F'(0) = 1. \end{cases}$$

Find a recursive formula for the coefficients of the nth Taylor polynomial for F at $x = 0$.

Solutions/Hints

48) Carefully look at the successive derivatives of each function at the indicated point.

49) Apply the Taylor polynomial formula.

50) Use the Lagrange Remainder Theorem to write $f(x) = (1+x)^{\frac{1}{3}}$ in terms of its first and second Taylor polynomials, and the remainder terms.

51) First we suppose that x_0 is a root of order k of the polynomial p. By definition there exists a polynomial r such that

$$p(x) = (x-x_0)^k r(x) \quad \text{and} \quad r(x_0) \neq 0;$$

we let n denote the degree of r. By Corollary 8.9, the Taylor polynomial

$$r(x_0) + r'(x_0)(x-x_0) + \frac{r''(x_0)}{2!}(x-x_0)^2 + \cdots + \frac{r^{(n)}(x_0)}{n!}(x-x_0)^n$$

of r at x_0 equals $r(x)$ itself, so that

$$p(x) = (x-x_0)^k r(x)$$

$$= (x-x_0)^k \left[r(x_0) + r'(x_0)(x-x_0) + \frac{r''(x_0)}{2!}(x-x_0)^2 + \cdots + \frac{r^{(n)}(x_0)}{n!}(x-x_0)^n \right]$$

$$= r(x_0)(x-x_0)^k + r'(x_0)(x-x_0)^{k+1} + \frac{r''(x_0)}{2!}(x-x_0)^{k+2} + \cdots + \frac{r^{(n)}(x_0)}{n!}(x-x_0)^{k+n}.$$

This shows that $p(x), p'(x), \ldots, p^{(k-1)}(x)$ are all polynomials in each of whose terms $(x-x_0)$ occurs to at least the first power. Thus

$$p(x_0) = p'(x_0) = \cdots = p^{(k-1)}(x_0) = 0,$$

and $p^{(k)}(x) =$

$$\frac{d^k}{dx^k}\left[r(x_0)(x-x_0)^k + r'(x_0)(x-x_0)^{k+1} + \frac{r''(x_0)}{2!}(x-x_0)^{k+2} + \cdots + \frac{r^{(n)}(x_0)}{n!}(x-x_0)^{k+n} \right] =$$

$$k!r(x_0) + [\text{terms containing } (x-x_0)],$$

so that

$$p^{(k)}(x_0) = k!r(x_0) \neq 0,$$

as desired. ■

52) Using what's given, how can you write the n^{th} derivative F?

Table of Contents

Math410: Homework

Name: UID: Section #:

Sequences And Series Of Numbers (9.1), Pointwise Convergence Of Sequences Of Functions (9.2), Uniform Convergence Of Sequences Of Functions (9.3)

53) Prove that the sequence $\left\{\frac{1}{n}+4\right\}$ is Cauchy by definition.

54) For the series $\sum_{k=1}^{\infty} a_k$, suppose that there is a number r with $0 \leq r < 1$ and a natural number N such that $|a_k|^{1/k} < r$ for all indices $k \geq N$. Prove that $\sum_{k=1}^{\infty} a_k$ converges absolutely.

55) Suppose that $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ are series of positive numbers such that $\lim_{k\to\infty}\left(\frac{a_k}{b_k}\right)=\ell$ with $\ell > 0$. Prove that the series $\sum_{k=1}^{\infty} a_k$ converges if and only if the series $\sum_{k=1}^{\infty} b_k$ converges.

56) For each natural number n and each number $x \geq 2$, define $f_n(x) = \frac{1}{1+x^n}$. Find the function $f : [2,\infty) \to \mathbb{R}$ to which the sequence $\{f_n : [2,\infty) \to \mathbb{R}\}$ converges pointwise.

57) For each natural number n and each number x in $[0,1]$, define

$$f_n(x) = \frac{x}{nx+1}$$

a) Find the function $f : [0,1] \to \mathbb{R}$ to which the sequence $\{f_n : [0,1] \to \mathbb{R}\}$ converges pointwise.

b) Prove that the convergence is uniform.

Solutions/Hints

53) **Proof:** Let $\epsilon > 0$ be given, choose $N > \frac{2}{\epsilon}$ by the AP property. Then, for all $m, n \geq N$, we have:

$$|a_n - a_m| = \left|\left(\frac{1}{n} + 4\right) - \left(\frac{1}{m} + 4\right)\right| = \left|\frac{1}{n} - \frac{1}{m}\right| \leq \left|\frac{1}{n}\right| + \left|\frac{1}{m}\right| \leq \frac{1}{N} + \frac{1}{N} = \frac{2}{N} < \epsilon$$

Thus, the sequence is Cauchy by definition.

54) **Proof:** Note that $|a_k|^{1/k} < r$ can be rewritten as $|a_k| \leq r^k$ for all natural numbers $k \geq N$, so that $\sum_{k=1}^{\infty} |a_k|$ converges by comparison with the geometric series $\sum_{k=1}^{\infty} r^k$.

55) **Proof:** Assume that $\lim_{k\to\infty} \left(\frac{a_k}{b_k}\right) = \ell > 0$. Additionally, assume that $\sum_{k=1}^{\infty} a_k$ converges. Since $\lim_{k\to\infty} \left(\frac{a_k}{b_k}\right) = \ell$, this implies that the sequence $\left\{\frac{1}{a_k/b_k} = \frac{b_k}{a_k}\right\}$ also converges. Since every convergent sequence is bounded, there exists $M \in \mathbb{R}$ with $\frac{b_k}{a_k} < M$ or that $b_k < Ma_k$ for all $k \in \mathbb{N}$. Since $\sum_{k=1}^{\infty} a_k$ converges, then by linearity $M \sum_{k=1}^{\infty} a_k$ also converges. Finally, using the Comparison Test implies that $\sum_{k=1}^{\infty} b_k$ converges also.

Conversely, assume that $\sum_{k=1}^{\infty} b_k$ converges. Since $\lim_{k\to\infty} \left(\frac{a_k}{b_k}\right) = \ell$, then $\left\{\frac{a_k}{b_k}\right\}$ is convergent and thus bounded. So there exists $M^* \in \mathbb{R}$ such that $\frac{a_k}{b_k} < M^*$, or $a_k < M^* b_k$, for all $k \in \mathbb{N}$. Since $\sum_{k=1}^{\infty} M^* b_k = M^* \sum_{k=1}^{\infty} b_k$ converges by linearity, then $\sum_{k=1}^{\infty} a_k$ converges as well by the Comparison Test.

■

56) This sequence converges pointwise to the zero function on $[2, \infty)$.

57) a) Let $x \in [0, 1]$. If $x = 0$ then $f_n(0) = 0$, for all $n \in \mathbb{N}$. For $x \in (0, 1]$, we have $f_n(x) = \frac{x}{nx + 1} = \frac{1}{n + \frac{1}{x}} < \frac{1}{n}$. But from previous work in the semester, we know that the sequence whose terms are all 0 and the sequence $\{\frac{1}{n}\}$ both

converge to 0. So the sequence of functions $f_n(x)$ converges pointwise to $f(x) = 0$ for all x on $[0, 1]$.

b) **Proof:** To prove uniform convergence, note that if $n \in \mathbb{N}$ and $x \in [0, 1]$, then

$$|f_n(x) - f(x)| = \left|\frac{x}{nx+1} - 0\right| = \frac{x}{nx+1}$$

But using our work above and the Archimedean Property, we can choose $N \in \mathbb{N}$ such that $1/N < \epsilon$. In turn, if $n \geq N$, then $|f_n(x) - f(x)| < \epsilon$, in *all* cases of $x \in [0, 1]$. Thus $\{f_n\}$ converges to f uniformly on $[0, 1]$.

■